URiCA 2025

Upcoming Researchers in Commutative Algebra

University of Nebraska

Lincoln, Nebraska

May 3–4, 2025

Conference Notes

URiCA 2025

Continuing the mission of KUMUNUjr

Inspired by the success of the KUMUNU commutative algebra conferences (KU, MU, NU), KUMUNUjr brought together graduate students and postdocs studying commutative algebra in the midwest to share their research and expertise while building lasting collaborations and friendships.

Upcoming Researchers in Commutative Algebra (URiCA) is the next chapter for KUMUNUjr.

Contents

1 Lecture 1 — Perfect Ideals with Fixed Deviation & Type by Xianglong Ni . . . 1

2 Lecture 2 — Stability Patterns in free resolutions of symmetric ideals by Kartik Ganapathy . . . 9

3 Lecture 3 — The Lefschetz properties for some modules over polynomial rings by Bek Chase . . . 17

4 Lecture 4 — Modules of Finite Length & Finite Projective Dimension by Nawaj KC . . . 25

1 Lecture 1 — Perfect Ideals with Fixed Deviation & Type by Xianglong Ni

Let S be a regular local ring or a standard graded regular ring containing $\mathbb{Q}$. Typical examples are

$$S = \mathbb{C}[x_1, \dots, x_n],$$

a localization of such a polynomial ring, or its completion.

Definition 1.1. Let $I \subseteq S$ be an ideal, and let $\mathbb{F}$ be a minimal free resolution of S/I. We say that I is **perfect** if the dual complex $\mathbb{F}^* = \operatorname{Hom}_S(\mathbb{F}, S)$ is acyclic except in the expected top homological degree.

Equivalently, if

$$\mathbb{F}\colon 0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = S \longrightarrow S/I \longrightarrow 0$$

is the minimal free resolution of S/I, then the dual complex has the form

$$0 \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_c^* \longrightarrow \operatorname{Ext}_S^c(S/I, S) \longrightarrow 0,$$

and acyclicity of $\mathbb{F}^*$ means that the only nonzero cohomology occurs at $\operatorname{Ext}_S^c(S/I, S)$. Let

$$c = \operatorname{pd}_S(S/I) = \operatorname{grade} I = \operatorname{codim} I.$$

Since S is regular, it is Cohen–Macaulay. Hence, by the Auslander–Buchsbaum formula,

$$I \text{ is perfect} \iff S/I \text{ is Cohen–Macaulay.}$$

Remark 1.2. For a quotient S/I over a regular local or standard graded ring, perfection is the condition that the projective dimension of S/I equals the height of I. Since S is Cohen–Macaulay, this is equivalent to S/I being Cohen–Macaulay.

Define the Betti numbers of S/I by

$$\beta_i = \beta_i(S/I) = \operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^S(S/I, k).$$

We define two numerical invariants:

- the **deviation**

$$d = \beta_1 - c;$$

- the **type**

$$t = \beta_c.$$

Example 1.3. The extreme and classical cases are:

- $d = 0$: I is a complete intersection;
- $t = 1$: S/I is Gorenstein;
- $d = 1$: I is an almost complete intersection.

Question 1.4. Can one upper bound the complexity of perfect ideals after fixing the triple (c, d, t)? More concretely, after fixing codimension c, deviation d, and type t, can one bound the intermediate Betti numbers

$$\beta_i(S/I), \qquad 2 \le i \le c - 1?$$

Example 1.5. For $c = 4$, $d = 2$, and $t = 2$, the Betti sequence has the form

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (1, 6, \beta_2, \beta_3, 2).$$

The question is whether β_2 and β_3 are bounded in this class.

If $d = 0$, then I is a complete intersection and $\mathbb{F}$ is isomorphic to the Koszul complex on a regular sequence of length c. In this case $t = 1$, and

$$\beta_i = \binom{c}{i}.$$

Observation 1.6. For every finite free resolution of S/I, the alternating sum of the total Betti numbers satisfies

$$\sum_{i=0}^{c} (-1)^i \beta_i = 0.$$

Remark 1.7. The equality above follows by taking ranks in the exact complex

$$0 \to F_c \to \cdots \to F_1 \to F_0 \to S/I \to 0.$$

Since S/I has rank 0 whenever $I \neq 0$, the alternating sum of the ranks of the free modules is zero.

The low-codimension cases are largely formal.

- If $c = 2$, then a perfect ideal has a Hilbert–Burch resolution

  $$0 \longrightarrow S^t \longrightarrow S^{2+d} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

  Using $\sum(-1)^i\beta_i = 0$, we get

  $$1 - (2 + d) + t = 0,$$

  hence

  $$t = d + 1.$$

- If $c = 3$, the Betti sequence is determined once d and t are fixed:

  $$0 \longrightarrow S^t \longrightarrow S^{2+d+t} \longrightarrow S^{3+d} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

  Indeed, the alternating-rank condition gives

  $$1 - (3 + d) + \beta_2 - t = 0,$$

  so

  $$\beta_2 = 2 + d + t.$$

- The first interesting case occurs in codimension 4. If $c = 4$ and $t = 1$, then S/I is Gorenstein, and the minimal free resolution is self-dual:

  $$\mathbb{F}^* \cong \mathbb{F}.$$

  Thus a resolution has the form

  $$0 \longrightarrow S \longrightarrow S^{\beta_3} \longrightarrow S^{\beta_2} \longrightarrow S^{4+d} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

  Self-duality gives

  $$\beta_3 = 4 + d.$$

  The alternating-rank condition then gives

  $$1 - (4 + d) + \beta_2 - (4 + d) + 1 = 0,$$

  hence

  $$\beta_2 = 6 + 2d.$$

For $c = 4$ and $d = 1$, one has a resolution of the form

$$0 \longrightarrow S^t \longrightarrow S^{\beta_3} \longrightarrow S^{\beta_2} \longrightarrow S^5 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Linkage

Let $I \subseteq S$, and let

$$(\alpha_1, \dots, \alpha_c) \subseteq I$$

be generated by a regular sequence of length c. Define

$$J = (\alpha_1, \dots, \alpha_c) : I = \{x \in S \mid xI \subseteq (\alpha_1, \dots, \alpha_c)\}.$$

If also

$$I = (\alpha_1, \dots, \alpha_c) : J,$$

then I and J are called **directly linked**. We write

$$I \sim J.$$

Remark 1.8. An ideal I is said to be in the linkage class of a complete intersection, or **licci**, if there exists a sequence of direct links

$$I \sim I_1 \sim I_2 \sim \cdots \sim \text{complete intersection}.$$

If I is perfect and $I \sim J$ by a complete intersection of the same codimension, then J is also perfect.

Remark 1.9. This is one of the basic stability properties of linkage in a Gorenstein ambient ring. Since S is regular, it is Gorenstein, and direct linkage by a complete intersection preserves Cohen–Macaulayness and codimension.

Ferrand Mapping Cone

Let $\mathbb{F}$ be a free resolution of S/I, and let $\mathbb{K}$ be the Koszul resolution of $S/(\alpha_1, \dots, \alpha_c)$. Let E denote the free module with basis corresponding to the regular sequence $\alpha_1, \dots, \alpha_c$. Thus the Koszul complex has terms $\bigwedge^i E$.

A comparison map $\phi : \mathbb{K} \to \mathbb{F}$ may be drawn schematically as

$$\begin{tikzcd}
\mathbb{K}: & 0 \arrow[r] & \bigwedge^c E \arrow[r] \arrow[d, "\phi_c"] & \cdots \arrow[r] & E \arrow[r] \arrow[d, "\phi_1"] & S \arrow[r] \arrow[d, equal] & S/(\alpha_1, \dots, \alpha_c) \arrow[r] \arrow[d] & 0 \\
\mathbb{F}: & 0 \arrow[r] & F_c \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & S \arrow[r] & S/I \arrow[r] & 0.
\end{tikzcd}$$

Ferrand's mapping cone construction says that the dual of the mapping cone resolves the residual ideal $J = (\alpha_1, \ldots, \alpha_c) : I$. Symbolically,

$$\mathrm{Cone}(\phi)^* \quad \text{resolves} \quad S/J.$$

At the level of free modules, this gives a complex of the form

$$S \longleftarrow F_c^* \oplus E^* \longleftarrow \cdots \longleftarrow F_2^* \oplus \bigwedge^2 E^* \longleftarrow F_1^* \longleftarrow 0.$$

Remark 1.10. The displayed modules are schematic: depending on the precise indexing convention for the mapping cone, the i-th free module in the dual cone is a direct sum of a dual term from $\mathbb{F}$ and a Koszul term. The key point used in the lecture is that the Betti numbers of the linked ideal can be estimated from the Betti numbers of I and the Koszul Betti numbers $\binom{c}{i}$.

For $c = 4$ and $d = 1$, suppose

$$0 \longrightarrow S^t \longrightarrow S^{\beta_3} \longrightarrow S^{\beta_2} \longrightarrow S^5 \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

and suppose $I \sim J$, where J is Gorenstein. A codimension-four Gorenstein resolution has the form

$$0 \longrightarrow S \longrightarrow S^n \longrightarrow S^{2n-2} \longrightarrow S^n \longrightarrow S \longrightarrow S/J \longrightarrow 0.$$

The Koszul resolution of the complete intersection $(\alpha_1, \ldots, \alpha_4)$ is

$$0 \longrightarrow S \longrightarrow S^4 \longrightarrow S^6 \longrightarrow S^4 \longrightarrow S \longrightarrow S/(\alpha_1, \ldots, \alpha_4) \longrightarrow 0.$$

By the mapping cone construction, one obtains schematically

$$S \longleftarrow S^5 \longleftarrow S^{n+t} \longleftarrow S^{2n+2} \longleftarrow S^n \longleftarrow 0.$$

This yields the estimate

$$\beta_2 \leq t + 10.$$

Remark 1.11. The inequality $\beta_2 \leq t + 10$ comes from comparing the ranks appearing in the mapping cone with the Koszul ranks in codimension 4, namely $1, 4, 6, 4, 1$. The additional 10 is the contribution forced by the codimension-four Koszul part.

Known Bounds and Maximal Betti Numbers

- For $c = 5$ and $t = 1$, the case $(d, t) = (1, 1)$ does not occur, by a result of Kunz.
- For $c = 5$, $d = 2$, and $t = 1$, Lopez proved that a licci ideal with these parameters has

one of the following Betti sequences:

$$\beta = (1, 7, 16, 16, 7, 1),$$

corresponding to a hypersurface section of a codimension-three ideal with five generators, or

$$\beta = (1, 7, 21, 21, 7, 1),$$

corresponding to Huneke–Ulrich examples.

There are no known non-licci examples of perfect ideals with

$$(d, t) = (2, 1).$$

Theorem 1.12 (Huneke–Vasconcelos, 1996)**.** *Suppose $(d, t) = (2, 1)$, and suppose S/I is generically a complete intersection and unobstructed. This applies, for example, to licci ideals. Then one can explicitly construct an exact sequence*

$$S^{\binom{c+3}{2}} \longrightarrow S^{c+2} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

In particular,

$$\beta_2 = \beta_{c-2} \leq \binom{c+3}{2}.$$

- If $(c, d, t) = (6, 2, 1)$, then
$$\beta = (1, 8, \leq 36, \leq 36, 8, 1).$$
Fact (Macaulay2 computation)*.* There is an ideal attaining these maximums.
- If $(c, d, t) = (7, 2, 1)$, then
$$\beta = (1, 9, \leq 45, -, -, 9, 1).$$
- If $(c, d, t) = (4, 2, 2)$, then
$$\beta = (1, 6, -, -, 2).$$

For the latter two cases, there exist ideals with arbitrarily large intermediate Betti numbers.

Proof sketch. The construction proceeds by repeated linkage.

(i) Start with a suitable homogeneous ideal
$$I_0 \subsetneq \mathbb{C}[x_1, \dots, x_n], \qquad n \geq c.$$

(ii) Define recursively
$$I_j = (\alpha_1, \dots, \alpha_c) : I_{j-1},$$

where $\alpha_1, \ldots, \alpha_c$ is chosen to be a regular sequence of maximum degree among the minimal generators of I_{j-1}.

This process yields

$$\lim_{N \to \infty} \beta_3(S/I_N) = \infty.$$

□

Remark 1.13. The point of choosing a regular sequence of maximum degree is to force the mapping cone in each linkage step to introduce increasingly large degree shifts. These degree shifts can produce unbounded intermediate Betti numbers even when the codimension, deviation, and type remain fixed.

Herzog Classes

Let

$$S = \mathbb{C}[x_1, \ldots, x_n], \qquad n \geq c.$$

Define P_c, a class of ideals in S, as follows. First,

$$(l_1, \ldots, l_c) \in P_c,$$

where $l_1, \ldots, l_c$ are independent linear forms. Now suppose $I \in P_c$ has minimal free resolution

$$0 \longrightarrow \bigoplus_{j=1}^{t} S(-b_{cj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{c+d} S(-b_{1j}) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Define

$$\varkappa = \frac{\sum_{j=1}^{t}(b_{cj} - c + 2)}{t+1} = \frac{1 + \sum_{j=1}^{c+d}(b_{1j} - 1)}{d+1}.$$

For each first-syzygy degree b_{1i}, set

$$\lambda_i = \varkappa + 1 - b_{1i},$$

and for each last-syzygy degree b_{ci}, set

$$N_i = \varkappa + c - 2 - b_{ci}.$$

Suppose

$$I = (h_1, \ldots, h_N), \qquad \deg h_i = \varkappa + 1 - b_{1i}.$$

If

$$\alpha_1, \ldots, \alpha_c \in \{h_1, \ldots, h_N\}$$

form a regular sequence, then

$$(\alpha_1, \ldots, \alpha_c) : I \in P_c.$$

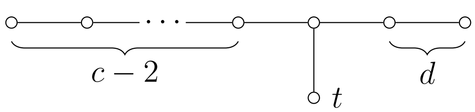
Conjecture 1.14. *The following statements are expected.*

(1) *If $I \in P_c$, then the Herzog class of S/I, and in particular the Betti numbers $\beta_i(S/I)$, is determined by the data (λ, N).*

(2) *Every codimension-c licci ideal is in the same Herzog class as some ideal $I \in P_c$.*

(3) *If $I_1, I_2 \in P_c$ have different data (λ, N), then they lie in different Herzog classes.*

Conjectures (1), (2), and (3) are known for $c = 3$. Conjectures (1) and (2) are known if the following associated graph is of ADE type.



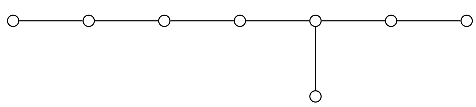
is ADE.

Examples from (c, d, t)-Parameters

Example 1.15. The parameter triple

$$(c, d, t) = (6, 2, 1)$$

yields the E_8 diagram.

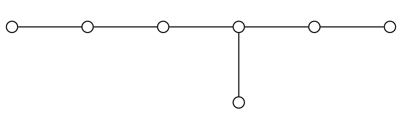


E_8

Example 1.16. The parameter triple

$$(c, d, t) = (7, 2, 1)$$

yields a diagram that is not Dynkin. It is denoted informally in the notes by "E_{10}".

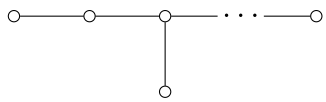


"E_{10}": not a Dynkin diagram

Example 1.17. The parameter family

$$(c, d, t) = (4, d, 1)$$

yields unspecified graphs in the lecture notes.

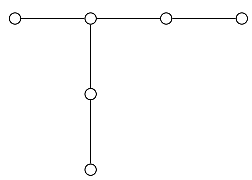


graphs depending on d

Example 1.18. The parameter triple

$$(c, d, t) = (4, 2, 2)$$

yields a diagram that is not Dynkin; it is labeled $E_6^{(1)}$ in the notes.



$E_6^{(1)}$, not a Dynkin diagram

This corresponds to the Betti shape

$$\beta = (1, 6, -, -, 2).$$

2 Lecture 2 — Stability Patterns in free resolutions of symmetric ideals by Kartik Ganapathy

Let

$$R = \Bbbk[x_1, \dots, x_n],$$

with the natural action of the symmetric group S_n by permutation of the variables. The motivating example is the infinite monomial ideal

$$I_\infty = \langle x_i^2 x_j \mid 1 \le i, j \le 3 \rangle.$$

Theorem 2.1 (Murai). *Let*

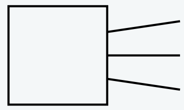
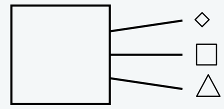
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

be a chain such that each $I_n \subseteq \Bbbk[x_1, \dots, x_n]$ is a monomial ideal stable under the natural action

$$S_n \curvearrowright \Bbbk[x_1, \dots, x_n].$$

Then there exists an integer N such that the Betti table of I_{N+1} is obtained from the

Betti table of I_N by a predictable extension along certain eventual lines. Symbolically, the phenomenon is represented as

$\beta(I_N) =$  *and* $\beta(I_{N+1}) =$  .

Betti table

Remark 2.2. The sketches above encode eventual stabilization of the shape of the Betti table under increasing the number of variables. The extra symbols $\diamond$, $\square$, and $\triangle$ indicate new Betti numbers appearing along predictable rays or lines in the Betti table. A central question is to identify these rays algebraically, rather than only combinatorially.

Problem 2.3. Interpret Murai's result algebraically. More precisely, given an ideal I, determine what algebraic structures contribute the lines of a fixed slope in the Betti table.

Finite Generation up to Symmetry

Theorem 2.4 (Cohen, 1967)**.** *Let*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

be a chain of homogeneous symmetric ideals, with

$$I_n \subseteq \Bbbk[x_1, \ldots, x_n].$$

Then there exist $N \in \mathbb{N}$ and finitely many elements

$$f_1, \ldots, f_r \in I_N$$

such that, for every $t \geq 0$,

$$I_{N+t} = \langle S_{N+t} \cdot f_i \mid 1 \leq i \leq r \rangle .$$

Remark 2.5. Here $S_{N+t} \cdot f_i$ denotes the orbit of f_i under the action of the symmetric group S_{N+t}. Thus Cohen's theorem says that a symmetric chain of ideals is eventually generated by finitely many orbits.

Given such a chain, define the two-variable Hilbert-series generating function

$$H_I(s,t) = \sum_{n \geq 1} H_{I_n}(t) s^n.$$

By work of Nagel–Römer, this series is a rational function in s and t. Nagpal–Snowden have studied S_∞-stable ideals in

$$\Bbbk[x_1, x_2, \ldots, x_n, \ldots]$$

and also S_∞-equivariant modules over

$$\Bbbk[x_1, \ldots, x_n, \ldots].$$

Conjecture 2.6 (Le–Nagel–Nguyen–Römer). *Let $\{I_n\}_{n\geq 1}$ be a suitable chain of homogeneous symmetric ideals. Then:*

(i) *there exists $a \in \mathbb{N}$ such that*

$$\operatorname{pdim}(I_n) = n - a \qquad \textit{for } n \gg 0;$$

(ii) *there exist $b, c \in \mathbb{N}$ such that*

$$\operatorname{reg}(I_n) = bn + c \qquad \textit{for } n \gg 0.$$

By the Auslander–Buchsbaum formula,

$$\operatorname{pdim}_{R_n}(I_n) + \operatorname{depth}_{R_n}(I_n) = n.$$

Hence, if $\operatorname{pdim}(I_n) = n - a$ for $n \gg 0$, then

$$\operatorname{depth}(I_n) = a \qquad \text{for } n \gg 0.$$

Guiding Idea

For fixed $i \geq 0$, one wants to endow the family

$$\left\{\operatorname{Ext}^i_{R_n}(\Bbbk, I_n)\right\}_{n\geq 1}$$

with an algebraic structure carrying an $\mathbb{N}$-grading and then prove finite-generation results for that structure. Equivalently, or dually, one seeks an analogous structure on the family of local cohomology modules

$$\left\{H^i_{\mathfrak{m}_n}(I_n)\right\}_{n\geq 1}.$$

GL_∞-Equivariant Modules

Definition 2.7. Let $\Bbbk$ be an infinite field of characteristic $p > 0$. Set

$$V = \Bbbk^\infty = \langle e_1, e_2, \ldots, e_n, \ldots \rangle$$

and

$$\mathrm{GL} = \mathrm{GL}_\infty = \mathrm{GL}(\Bbbk^\infty).$$

Let

$$R = \mathrm{Sym}(V) = \Bbbk[x_1, x_2, \ldots].$$

A GL_∞-equivariant R-module is an R-module M, equipped with an action of GL_∞, such that:

(i) for all $r \in R$, $m \in M$, and $g \in \mathrm{GL}$,

$$g(rm) = g(r) \cdot g(m);$$

equivalently, the multiplication map

$$R \otimes M \longrightarrow M$$

is GL-equivariant;

(ii) M is a smooth representation of GL_∞. In characteristic p, the relevant polynomial/smooth GL_∞-representations have subquotients related to Schur functors.

For such an M, one defines

$$M_n = M^{\mathrm{GL}_{\infty - n}},$$

and M_n is naturally an R_n-module. Thus the correspondence is schematically

$$M \longleftrightarrow M_n = M^{\mathrm{GL}_{\infty - n}},$$

where M_n is a module over $R_n = \Bbbk[x_1, \ldots, x_n]$.

Remark 2.8. The passage from M to $M_n = M^{\mathrm{GL}_{\infty - n}}$ extracts the finite-variable part of the infinite equivariant module. This is the mechanism by which one packages a sequence of R_n-modules into a single equivariant object.

Torsion Functors and Local Cohomology in Positive Characteristic

Definition 2.9. Fix $e \in \mathbb{N}$. Define a functor

$$\Gamma_e : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$$

by

$$\Gamma_e(M) = \left\{ x \in M \;\middle|\; (\mathfrak{m}^{[p^e]})^n x = 0 \text{ for } n \gg 0 \right\}.$$

Equivalently, $\Gamma_e(M)$ is the zeroth local cohomology of M with respect to the Frobenius power $\mathfrak{m}^{[p^e]}$.

Remark 2.10. The definition is meaningful because the ideals $\mathfrak{m}^{[p^e]}$ and $\mathfrak{m}^{[p^d]}$ are not cofinal when $d \neq e$.

Definition 2.11. Define another torsion functor

$$\Gamma : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$$

by

$$\Gamma(M) = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{there exists a nonzero } \mathrm{GL}\text{-stable ideal } I \subseteq R \\ \text{such that } I^n x = 0 \text{ for } n \gg 0 \end{array} \right\}.$$

Theorem 2.12. *Let M be a finitely generated* GL*-equivariant R-module. Then, for every $e \in \mathbb{N}$, the right derived functor*

$$R\Gamma_e(M)$$

is also a finitely generated GL*-equivariant R-module.*

Remark 2.13. The notes also indicated an analogous statement involving $R\Gamma(M)$.

Slopes in Betti Tables

Theorem 2.14. *Let M be a finitely generated* GL*-equivariant R-module.*

(i) *The nonlinear lines occurring in the Betti table $\beta(M)$ all have slopes of the form*

$$p^e - 1,$$

where e is allowed to vary.

(ii) *The complex $R\Gamma_e(M)$ has a pure slope $p^e - 1$ resolution, and this resolution agrees with that of M eventually.*

Remark 2.15. This gives an algebraic interpretation of the lines appearing in eventual Betti tables: they arise from derived torsion functors associated to Frobenius powers of the irrelevant ideal.

Koszul Algebras and Weighted Rational Curves

Definition 2.16. A $\mathbb{Z}$-graded $\Bbbk$-algebra R is called *Koszul* if the residue field $R/\mathfrak{m} \cong \Bbbk$ has a linear free resolution over R. Equivalently, the minimal graded free resolution of $\Bbbk$ over R has the form

$$0 \longleftarrow \Bbbk \longleftarrow R \longleftarrow R(-1)^{b_1} \longleftarrow R(-2)^{b_2} \longleftarrow \cdots .$$

Weighted Rational Curves

Example 2.17. Consider a weighted rational curve of type $(3, 2)$. Let

$$D = [0 : 1] \in \mathbb{P}^1.$$

The map is

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}(1, 1, 1, 2, 2),$$

given by

$$[s, t] \longmapsto \left[\underbrace{s^3 : s^2t : st^2}_{\substack{\text{degree } d \\ \text{monomials} \\ \text{vanishing at} \\ D}} : \underbrace{st^5 : t^6}_{\substack{\text{rest of} \\ \text{degree } d.e. \\ \text{monomials}}} \right].$$

Let

$$S = \Bbbk[x_0, x_1, x_2, y_0, y_1].$$

Define a homomorphism

$$\phi : S \longrightarrow \Bbbk[s, t]$$

by

$$\begin{aligned} x_0 &\longmapsto s^3, \\ x_1 &\longmapsto s^2t, \\ x_2 &\longmapsto st^2, \\ y_0 &\longmapsto st^5, \\ y_1 &\longmapsto t^6. \end{aligned}$$

Set

$$I = \ker\phi \qquad \text{and} \qquad R = S/I.$$

$$S = \mathbb{k}[x_0, x_1, x_2, y_0, y_1] \xrightarrow{\ \phi\ } \mathbb{k}[s, t]$$

$$x_0 \longmapsto s^3$$

$$x_1 \longmapsto s^2t$$

$$x_2 \longmapsto st^2$$

$$y_0 \longmapsto st^5$$

$$y_1 \longmapsto t^6$$

Facts

(⊙) The ideal I is determinantal:

$$I = I_2\begin{pmatrix} x_0 & x_1 & x_2^2 & y_0 \\ x_1 & x_2 & y_0 & y_1 \end{pmatrix}.$$

(⊙) The quotient ring $R = S/I$ is Cohen–Macaulay.

Question about the Koszul Property

For the residue field $\Bbbk = R/\mathfrak{m}$, the beginning of a graded free resolution over R has the form

$$0 \longleftarrow \Bbbk \longleftarrow R \longleftarrow \begin{matrix} R(-1)^3 \\ \oplus \\ R(-2)^2 \end{matrix} \longleftarrow \cdots .$$

This is not a linear resolution in the standard sense because generators occur in degrees 1 and 2.

Definition 2.18 (Herzog–Reiner–Welker)**.** A $\mathbb{Z}$-graded $\Bbbk$-algebra R is called *nonstandard Koszul* if the associated graded ring

$$\operatorname{gr}_{\mathfrak{m}} R$$

is Koszul.

Recall that

$$\operatorname{gr}_{\mathfrak{m}} R = \underbrace{R/\mathfrak{m}}_{0} \oplus \underbrace{\mathfrak{m}/\mathfrak{m}^2}_{1} \oplus \underbrace{\mathfrak{m}^2/\mathfrak{m}^3}_{2} \oplus \cdots .$$

Example 2.19. Let

$$R = \Bbbk[x,y]/(y^2 - x^3).$$

Then

$$\operatorname{gr}_{\mathfrak{m}} R \cong \Bbbk[x,y]/(y^2),$$

which is Koszul. Hence R is nonstandard Koszul.

Example 2.20. Let

$$R = \Bbbk[x,y]/(y^2 - x^3 - x^6).$$

Then

$$\operatorname{gr}_{\mathfrak{m}} R \cong \Bbbk[x,y]/(y^2),$$

which is Koszul. Hence R is nonstandard Koszul. This ring is the coordinate ring of a weighted rational curve.

Question 2.21. Is the coordinate ring R of the weighted rational curve above nonstandard Koszul?

For this example, one computes

$$\operatorname{gr}_{\mathfrak{m}}(R) = \Bbbk[x_0, x_1, x_2, y_0, y_1] \Big/ I_2 \begin{pmatrix} x_0 & x_1 & 0 & y_0 \\ x_1 & x_2 & y_0 & y_1 \end{pmatrix}.$$

In fact, the minors of the displayed matrix form a quadratic Gröbner basis. Therefore $\mathrm{gr}_{\mathfrak{m}}(R)$ is Koszul, and hence R is nonstandard Koszul.

Theorem 2.22 (Davis–Sabionka, 2024)**.** *Let R be the coordinate ring of a weighted rational curve. Then:*

(1) *the defining ideal I is determinantal;*

(2) *the ring R is Cohen–Macaulay;*

(3) *the ring R is nonstandard Koszul.*

3 Lecture 3 — The Lefschetz properties for some modules over polynomial rings by Bek Chase

Definitions and Basics

Let $\Bbbk$ be a field of characteristic 0. Let

$$S = \Bbbk[x_1, \ldots, x_n]$$

be a standard graded polynomial ring. Let

$$A = S/I = \bigoplus_{i=0}^{c} A_i$$

be an Artinian graded algebra, and let

$$M = \bigoplus_{i=a}^{b} M_i$$

be an Artinian graded S-module.

Definition 3.1 (Hilbert function and Hilbert series)**.** The **Hilbert function** of M is the function

$$i \longmapsto \dim_{\Bbbk} M_i.$$

The **Hilbert series** of M is

$$\mathrm{HS}(M) = \sum_{i=a}^{b} \dim_{\Bbbk}(M_i) t^i.$$

Definition 3.2 (Weak Lefschetz property)**.** The module M has the **weak Lefschetz property**, abbreviated WLP, if there exists a linear form $\ell \in S_1$ such that the multiplication map

$$\times\ell \colon M_i \longrightarrow M_{i+1}$$

has maximal rank for every i. Equivalently, for each i, this map is either injective or surjective.

Definition 3.3 (Strong Lefschetz property)**.** The module M has the **strong Lefschetz property**, abbreviated SLP, if there exists a linear form $\ell \in S_1$ such that, for every $d \geq 1$, the multiplication map

$$\times\ell^d \colon M_i \longrightarrow M_{i+d}$$

has maximal rank for every i.

Fact. The following standard facts are used throughout the lecture.

(1) If M has monomial generators, then it is enough to test the Lefschetz properties using

$$\ell = x_1 + \cdots + x_n.$$

(2) If M has the WLP or the SLP, then $\mathrm{HS}(M)$ is unimodal.

Remark 3.4. For monomial quotients over a field of characteristic 0, the choice $\ell = x_1 + \cdots + x_n$ is justified by semicontinuity and the torus action on the variables. A general linear form has maximal rank in the relevant multiplication maps if and only if this symmetric linear form does.

Theorem 3.5 (Stanley, 1980)**.** *The monomial complete intersection*

$$\Bbbk[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$$

has the strong Lefschetz property.

Remark 3.6. The ideal $(x_1^{a_1}, \ldots, x_n^{a_n})$ is generated by a homogeneous regular sequence. Thus the quotient is a zero-dimensional complete intersection:

$$\dim S/(x_1^{a_1}, \ldots, x_n^{a_n}) = 0.$$

Main Questions

The lecture raises the following guiding questions.

Question 3.7. Do all complete intersections have the strong Lefschetz property?

Remark 3.8. For an Artinian algebra A, the socle is

$$\operatorname{soc}(A) = (0 : \mathfrak{m}),$$

where $\mathfrak{m} = A_+$ is the homogeneous maximal ideal. A standard graded Artinian complete intersection is Gorenstein, hence has one-dimensional socle. In other words, it has type 1. In embedding dimension $n = 4$, not all complete intersections have the strong Lefschetz property.

Question 3.9. Do all Gorenstein algebras have the strong Lefschetz property?

Remark 3.10. Again, in embedding dimension $n = 4$, the answer is no.

Question 3.11. What other algebras or modules have the strong Lefschetz property?

Macaulay Duality and Binomial Dual Generators

Consider Artinian algebras described by inverse systems. Let F be a Macaulay dual generator. Define

$$A_F = S/\operatorname{ann}(F),$$

where

$$\operatorname{ann}(F) = \{P \in S \mid P \circ F = 0\},$$

and $P \circ F$ denotes the action of P on F by differentiation.

Remark 3.12. In Macaulay inverse systems, the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ acts on a divided power or polynomial dual ring by partial differentiation. The annihilator $\operatorname{ann}(F)$ records precisely those differential operators killing F. The quotient $A_F = S/\operatorname{ann}(F)$ is Artinian whenever F has finite degree.

The lecture emphasizes the following cases.

(1) If F is a monomial, then A_F is a monomial complete intersection. Therefore A_F has the SLP by Stanley's theorem.

(2) If F is a binomial, then the structure of A_F becomes more subtle. The question is:

$$A_F \ = ?$$

Proposition 3.13 (ADFMMRSV). *Let $n = 3$. For every binomial F, the initial algebra $\mathrm{in}_<(A_F)$ is one of the following:*

(1) *a monomial complete intersection of type 2;*

(2) *an algebra whose defining initial ideal has the form*

$$(x^a, y^b, z^c, x^\alpha y^\beta z^\gamma);$$

(3) *an intersection of ideals of the forms appearing in (1) and (2).*

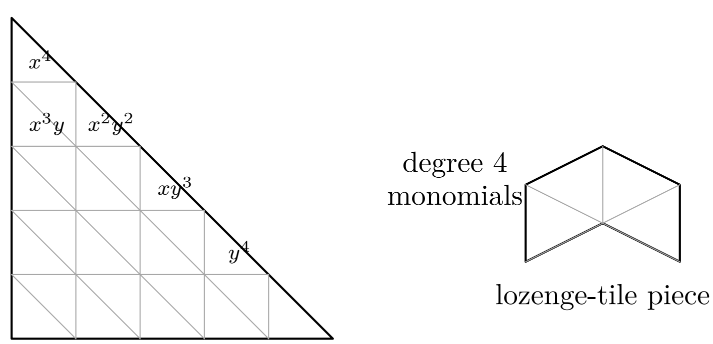
Remark 3.14. The point of passing to an initial algebra is that monomial data can often be analyzed combinatorially. In particular, Lefschetz maps for monomial quotients can be studied using lattice paths, determinants of binomial coefficient matrices, and tilings of triangular regions.

Biannular Regions and Central-Simple Modules

Let

$$I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma).$$

The weak Lefschetz property for ideals of this type is characterized by Cook II and Nagel using biannular regions and lozenge tilings.



Remark 3.15. A triangular monomial region encodes standard monomials in three variables. Removing subregions corresponding to monomial generators produces a punctured triangular region. The existence and enumeration of lozenge tilings of such regions can detect whether certain multiplication maps have maximal rank.

Central-Simple Modules

Let A be an Artinian graded $\Bbbk$-algebra, and let $\ell \in A_1$. Let

$$P = \min\{s \in \mathbb{Z}_{\geq 1} \mid \ell^s = 0\}.$$

There is a filtration

$$A = (0 : \ell^P) + (\ell) \supseteq (0 : \ell^{P-1}) + (\ell) \supseteq \cdots \supseteq (0 : \ell^0) + (\ell) = (\ell).$$

Equivalently, the filtration may be displayed as

$$(0 : \ell^P) + (\ell) \hookleftarrow (0 : \ell^{P-1}) + (\ell) \hookleftarrow \cdots \hookleftarrow (0 : \ell) + (\ell) \hookleftarrow (\ell).$$

Definition 3.16 (Central-simple module)**.** The i-th **central-simple module** of A with respect to ℓ is

$$V_{i,\ell} = \frac{(0 : \ell^i) + (\ell)}{(0 : \ell^{i-1}) + (\ell)}.$$

Let

$$\widetilde{V}_\ell = \bigoplus_{i=1}^{P} V_{i,\ell}.$$

Fact. The Hilbert series of $\widetilde{V}_\ell$ agrees with that of the associated graded algebra with respect to the ideal (ℓ), and hence with that of A:

$$\mathrm{HS}(\widetilde{V}_\ell) = \mathrm{HS}\Big(\mathrm{gr}_{(\ell)} A\Big) = \mathrm{HS}(A).$$

This equality is considered degree by degree for $0 \leq i \leq P$.

Theorem 3.17 (HW 2007; Chase 2024)**.** *The algebra A has the strong Lefschetz property if and only if there exists $\ell \in A_1$ such that $\widetilde{V}_\ell$ has the strong Lefschetz property.*

Remark 3.18. The central-simple modules decompose the behavior of A relative to the nilpotent operator $\times\ell$. Thus the strong Lefschetz property for A can be tested by studying the Lefschetz behavior of the graded pieces arising from this filtration.

Application to the Ideal $I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma)$

Let

$$I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma) \subseteq \Bbbk[x, y, z],$$

and set

$$A = S/I.$$

Take

$$\ell = z.$$

Then the central-simple modules of A with respect to z include the following modules:

$$V_{1,z} = \frac{(0 : z^1) + (z)}{(0 : z^0) + (z)} \cong \frac{\Bbbk[x, y]}{(x^a, y^b)}.$$

The next relevant central-simple module has the form

$$V_{2,z} \cong \frac{(x^\alpha, y^\beta)}{(x^a, y^b)}(-\alpha - \beta) \otimes_\Bbbk \frac{\Bbbk[z]}{(z^{\gamma_i})}.$$

Remark 3.19. Tensor products do not always preserve the strong Lefschetz property. For example, the tensor product of Hilbert functions

$$(1, 3, 3, 1) \otimes (1, 3, 1, 1)$$

does not automatically yield a module with the strong Lefschetz property.

Remark 3.20. The shift $(-\alpha - \beta)$ indicates a grading shift. The module

$$(x^\alpha, y^\beta)/(x^a, y^b)$$

is viewed as a graded $\Bbbk[x, y]$-module, while

$$\Bbbk[z]/(z^{\gamma_i})$$

records the finite z-direction coming from the z-adic filtration.

Theorem 3.21 (Chase 2024). *Let $I \subseteq \Bbbk[x, y]$ be a homogeneous Artinian ideal, and let*

$$J = (x^a, y^b) \subseteq I.$$

Then the quotient module I/J has the strong Lefschetz property.

Remark 3.22. The lecture notes record the following principle:

$$S/I \text{ has the SLP} \iff S/\operatorname{in}_<(I) \text{ has the SLP}.$$

Remark 3.23. Such equivalences must usually be interpreted with care. Strong Lefschetz behavior is not preserved under arbitrary Gröbner degeneration in complete generality. In the setting of the lecture, the point is that the relevant initial ideals are sufficiently controlled so that the Lefschetz problem may be transferred to a monomial or combinatorial model.

Lindström–Gessel–Viennot and Lattice Paths

The Lindström–Gessel–Viennot lemma relates determinants to families of nonintersecting lattice paths. In the present setting, the determinant of a matrix whose entries are binomial coefficients counts suitable lattice paths:

$$\det\begin{pmatrix}\text{matrix of}\\ \text{binomial coefficients}\end{pmatrix} = \text{enumeration of lattice paths.}$$

The connection with Lefschetz maps comes from the expansion

$$\ell^d = (x+y)^d = \binom{d}{0}x^d + \binom{d}{1}x^{d-1}y + \cdots + \binom{d}{d}y^d.$$

Thus multiplication by ℓ^d in a monomial basis is represented by a matrix of binomial coefficients.

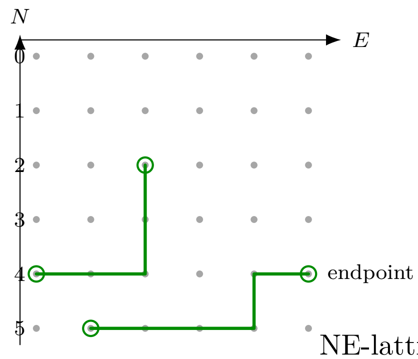
Example 3.24. One obtains matrices such as

$$\det\begin{pmatrix}\binom{3}{0} & \binom{3}{1}\\ \binom{3}{2} & \binom{3}{3}\end{pmatrix}.$$

The notes also record a larger binomial-coefficient matrix:

$$\begin{pmatrix}\binom{5}{2} & \binom{4}{2} & \binom{3}{2}\\ \binom{5}{4} & \binom{4}{4} & \binom{3}{4}\\ 0 & 0 & 0\end{pmatrix}.$$

Since the final row is zero, its determinant is 0.



Remark 3.25. A northeast lattice path consists of unit steps east and north. The number of paths from one lattice point to another is a binomial coefficient, because choosing the positions of the north steps among all steps determines the path. The Lindström–Gessel–Viennot lemma upgrades this single-path enumeration to a determinant formula for nonintersecting path families.

A Strong Lefschetz Criterion

Theorem 3.26 (Chase 2024). *Let*

$$I = \mathrm{in}_<(A_F)$$

be the initial ideal arising from a binomial Macaulay dual generator in three variables. The quotient S/I has the strong Lefschetz property if the following conditions hold:

(1)
$$\alpha + \beta \le a + b - c \le \alpha + \beta + 1.$$

(2)
$$\min(\alpha, \beta) \ne \max(\alpha, \beta) = \min(\alpha + \beta, a, b)$$

implies that S/I has a symmetric Hilbert series.

(3)
$$(\sim\sim\sim)$$

Theorem 3.27 (ADFMMRSV). *For any algebra A_F with $n = 3$, if F is binomial, then A_F has the strong Lefschetz property.*

Remark 3.28. The overall strategy is therefore:

$$F \text{ binomial} \longrightarrow A_F = S/\operatorname{ann}(F) \longrightarrow \operatorname{in}_<(A_F) \longrightarrow \substack{\text{monomial/}\\ \text{combinatorial}\\ \text{model}} \longrightarrow \text{SLP}.$$

The monomial model is then studied using central-simple modules, binomial-coefficient matrices, and lattice-path enumerations.

4 Lecture 4 — Modules of Finite Length & Finite Projective Dimension by Nawaj KC

The Basic Question

Question 4.1. How large can the annihilator of an R-module of finite projective dimension be?

Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $M \neq 0$ be an R-module. Recall that

$$\operatorname{ann}_R(M) = \{r \in R \mid rM = 0\}.$$

If $M \neq 0$ is finitely generated, then $\operatorname{ann}_R(M) \subseteq \mathfrak{m}$ whenever R is local and M is not annihilated by a unit.

Theorem 4.2 (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. If $M \neq 0$ is an R-module such that*

$$\operatorname{ann}_R(M) = \mathfrak{m} \quad \textit{and} \quad \operatorname{pdim}_R M < \infty,$$

then R is regular.

Remark 4.3. For a finitely generated R-module M, Nakayama's lemma gives

$$\mathfrak{m}M \subsetneq M \iff M/\mathfrak{m}M \neq 0 \iff M \otimes_R k \neq 0,$$

where $k = R/\mathfrak{m}$ is the residue field.

Theorem 4.4 (New Intersection Theorem)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. If*

$M \neq 0$ is an R-module such that

$$\operatorname{ann}_R(M) \supseteq \mathfrak{m}^n \quad \textit{for some } n \geq 1$$

and

$$\operatorname{pdim}_R M < \infty,$$

then R is Cohen–Macaulay.

Remark 4.5. The phrase "large annihilator" means that $\operatorname{ann}_R(M)$ contains a high-power neighborhood of the closed point. Equivalently, M is killed by a sufficiently large power of $\mathfrak{m}$, so M is supported only at $\mathfrak{m}$ and has finite length, provided M is finitely generated.

Quantifying the Size of the Annihilator

The goal is to quantify the previous theorems.

Definition 4.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be an R-module. Define

$$U_R(M) = \min\{i \mid \mathfrak{m}^i \subseteq \operatorname{ann}_R(M)\},$$

whenever this minimum exists.

Thus a small value of $U_R(M)$ means that the annihilator of M is large:

$$U_R(M) \text{ small} \iff \operatorname{ann}_R(M) \text{ large}.$$

Question 4.7. What is

$$\min\{U_R(M) \mid M \neq 0, \ \operatorname{pdim}_R M < \infty\}?$$

More precisely, among all nonzero R-modules of finite projective dimension, which ones have the largest annihilator?

Guiding Philosophy: Quotients by Systems of Parameters

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let

$$\underline{x} = x_1, \ldots, x_d$$

be a maximal R-regular sequence. Since R is Cohen–Macaulay, such a sequence is also a system of parameters. Consider the quotient

$$R/(\underline{x}) = R/(x_1, \ldots, x_d).$$

This module has two important properties.

(1) It has finite length. Indeed, since $\underline{x}$ is a system of parameters, the ideal $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary. Therefore there exists $n \geq 1$ such that

$$\mathfrak{m}^n \subseteq (x_1, \ldots, x_d).$$

Hence

$$\operatorname{ann}_R(R/(\underline{x})) = (\underline{x})$$

contains a power of $\mathfrak{m}$, so the annihilator is large.

(2) It has finite projective dimension. Since $\underline{x}$ is an R-regular sequence of length d, the Koszul complex on $\underline{x}$ is a free resolution of $R/(\underline{x})$. Hence

$$\operatorname{pdim}_R R/(\underline{x}) = d < \infty.$$

Thus quotients by systems of parameters are the simplest or smallest R-modules of finite length and finite projective dimension.

Remark 4.8. In a Cohen–Macaulay local ring, every system of parameters is a regular sequence. Therefore the Koszul complex

$$\operatorname{Kos}(x_1, \ldots, x_d)$$

is not merely a complex with zeroth homology $R/(\underline{x})$; it is a finite free resolution of $R/(\underline{x})$.

$$\operatorname{Kos}(x_1, \ldots, x_d) \xrightarrow{\simeq} R/(x_1, \ldots, x_d).$$

Three Conjectural Minimality Statements

Let $M \neq 0$ be an R-module such that

$$\ell_R(M) < \infty \quad \text{and} \quad \operatorname{pdim}_R M < \infty.$$

Since M has finite length, one has $U_R(M) < \infty$.

The guiding philosophy suggests that M should be at least as large as some quotient by a system of parameters. This leads to the following conjectural principles.

Conjecture 4.9 (Betti-number lower bound; B–E–M)**.** *Let $M \neq 0$ be an R-module with $\ell_R(M) < \infty$ and $\operatorname{pdim}_R M < \infty$. Then*

$$\beta_i^R(M) \geq \beta_i^R(R/(\underline{x}))$$

for some, hence any, system of parameters $\underline{x}$.

Remark 4.10. If R is Cohen–Macaulay of dimension d, then $R/(\underline{x})$ is resolved by the Koszul complex. Hence

$$\beta_i^R(R/(\underline{x})) = \binom{d}{i}.$$

Therefore the conjecture predicts the lower bound

$$\beta_i^R(M) \geq \binom{d}{i}.$$

Conjecture 4.11 (Length lower bound; I–M–N)**.** *Let $M \neq 0$ be an R-module with $\ell_R(M) < \infty$ and $\operatorname{pdim}_R M < \infty$. Then*

$$\ell_R(M) \geq \min\{\ell_R(R/(\underline{x})) \mid \underline{x} \text{ is a system of parameters}\}.$$

Remark 4.12. This conjecture is open in the local case. It is known for cyclic modules in dimension $\dim R \leq 2$, due to Nawaj and Andrew.

Conjecture 4.13 (Annihilator-order lower bound; C–H–P–R)**.** *Let $M \neq 0$ be an R-module with $\ell_R(M) < \infty$ and $\operatorname{pdim}_R M < \infty$. Then*

$$U_R(M) \geq \min\{U_R(R/(\underline{x})) \mid \underline{x} \text{ is a system of parameters}\}.$$

Remark 4.14. The annihilator-order conjecture is true when R is Gorenstein and $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay. This was proved by A–B–I–M in 2010. In this setting, one assumes that the residue field $R/\mathfrak{m}$ is infinite.

The Nawaj–Pollitz Theorem

Theorem 4.15 (Nawaj–Pollitz, 2025)**.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with infinite residue field. If the associated graded ring*

$$\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is Cohen–Macaulay, then the annihilator-order conjecture holds. That is, for every nonzero R-module M with $\ell_R(M) < \infty$ and $\mathrm{pdim}_R\, M < \infty$, one has

$$U_R(M) \geq \min\{U_R(R/(\underline{x})) \mid \underline{x} \text{ is a system of parameters}\}.$$

Remark 4.16. If
$$R = k[[x_1, \ldots, x_n]]_{\mathfrak{m}}/(f_1, \ldots, f_t),$$
then the condition that $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay implies that R is Cohen–Macaulay.

Remark 4.17. The implication $\mathrm{gr}_{\mathfrak{m}}(R)$ Cohen–Macaulay $\Rightarrow$ R Cohen–Macaulay is standard. The associated graded ring records the leading forms of elements with respect to the $\mathfrak{m}$-adic filtration. If the associated graded ring is Cohen–Macaulay, then its depth is maximal, and this forces the original local ring to be Cohen–Macaulay.

Assume now that $R/\mathfrak{m}$ is infinite. Then one can choose sufficiently general elements

$$\underline{x} = x_1, \ldots, x_d \in \mathfrak{m} \setminus \mathfrak{m}^2$$

such that their initial forms
$$\underline{x}^* = x_1^*, \ldots, x_d^* \in \mathrm{gr}_{\mathfrak{m}}(R)$$
form a regular sequence in $\mathrm{gr}_{\mathfrak{m}}(R)$.

Remark 4.18. The condition $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ means that each x_i has order one. The regularity of the initial forms $x_1^*, \ldots, x_d^*$ in $\mathrm{gr}_{\mathfrak{m}}(R)$ gives strong control over the $\mathfrak{m}$-adic order of elements modulo $(x_1, \ldots, x_d)$. This is the technical bridge between the associated graded hypothesis and the annihilator-order bound.

The Key Lemma

Let
$$A = R/(\underline{x}).$$

Since $\underline{x}$ is a system of parameters in a Cohen–Macaulay local ring, A is an Artinian local ring. Choose a socle element

$$S \in A$$

of maximal $\mathfrak{m}$-adic order. Thus

$$S \in \mathfrak{m}^n A \quad \text{but} \quad S \notin \mathfrak{m}^{n+1} A,$$

where

$$n = \operatorname{ord}(S).$$

Since S lies in the socle, one has

$$\mathfrak{m} S = 0.$$

In particular, S represents a copy of the residue field k inside A.

The lecture diagram may be organized as follows:

$$1 \longmapsto S$$

$$\begin{tikzcd}
k \arrow[r, hook, "\iota"'] & R/(x_1, \dots, x_d) \\
F \arrow[u, "\simeq"'] \arrow[r, "\sigma"] & K \arrow[u, "\simeq"]
\end{tikzcd}$$

Here $K = \operatorname{Kos}(x_1, \dots, x_d)$ is the Koszul resolution of $R/(\underline{x})$, F is a free resolution of k, and $\sigma : F \to K$ is a chain map lifting the inclusion

$$\iota : k \hookrightarrow R/(\underline{x}), \qquad 1 \longmapsto S.$$

Lemma 4.19 (Key Lemma)**.** *Let $S \in R/(\underline{x})$ be a socle element of maximal order $n = \operatorname{ord}(S)$, and let*

$$\sigma : F \longrightarrow K$$

be a chain map lifting the inclusion $k \hookrightarrow R/(\underline{x})$ determined by $1 \mapsto S$. Then

$$\sigma(F) \subseteq \mathfrak{m}^n K.$$

Equivalently, each component

$$\sigma_i : F_i \longrightarrow K_i$$

has image contained in $\mathfrak{m}^n K_i$.

> *Remark* 4.20. The point of the key lemma is that the chain map σ is not arbitrary. Because the element S has high $\mathfrak{m}$-adic order, the map lifting $1 \mapsto S$ can be chosen so that all components of the chain map land inside the same high power $\mathfrak{m}^n$ of the maximal ideal.

Since S has maximal order n in the socle of $A = R/(\underline{x})$, one obtains

$$U_R(R/(\underline{x})) = n + 1.$$

Indeed, the equality reflects the fact that the last nonzero $\mathfrak{m}$-adic layer of $R/(\underline{x})$ occurs in order n, so $\mathfrak{m}^{n+1}$ kills the quotient while $\mathfrak{m}^n$ does not.

Tensoring the Diagram with a Finite-Length Module

Let M be an R-module. Consider the following diagram of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & F_{d+1} \arrow[r] \arrow[d] & F_d \arrow[r] \arrow[d, "\sigma_d"] & \cdots \arrow[r] & F_0 \arrow[r] \arrow[d, "\sigma_0"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & K_d \arrow[r] & \cdots \arrow[r] & K_0 \arrow[r] & 0.
\end{tikzcd}$$

Tensoring with M gives

$$\sigma \otimes_R M : F \otimes_R M \longrightarrow K \otimes_R M.$$

Assume that

$$U_R(M) \leq n.$$

Equivalently,

$$\mathfrak{m}^n \subseteq \operatorname{ann}_R(M).$$

Since the key lemma gives $\sigma(F) \subseteq \mathfrak{m}^n K$, it follows that every component of $\sigma \otimes_R M$ is zero. Thus we obtain a diagram

$$\begin{tikzcd}
F_{d+1} \otimes_R M \arrow[r] \arrow[d] & F_d \otimes_R M \arrow[r] \arrow[d, "\sigma_d \otimes_R M"'] & \cdots \arrow[r] & F_0 \otimes_R M \arrow[r] \arrow[d, "\sigma_0 \otimes_R M"] & 0 \\
0 \arrow[r] & K_d \otimes_R M \arrow[r] & \cdots \arrow[r] & K_0 \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

in which

$$\sigma \otimes_R M = 0.$$

Therefore

$$\mathrm{Cone}(\sigma \otimes_R M) = \mathrm{Cone}(F \otimes_R M \xrightarrow{0} K \otimes_R M) \cong K \otimes_R M \oplus \Sigma(F \otimes_R M),$$

where Σ denotes suspension of complexes. Moreover,

$$\mathrm{Cone}(\sigma) \simeq_R R/(\underline{x}, S).$$

Hence

$$\mathrm{Cone}(\sigma \otimes_R M) = \mathrm{Cone}(\sigma) \otimes_R M.$$

Taking homology gives

$$H_{d+1}\Big(\mathrm{Cone}(\sigma) \otimes_R M\Big) \cong H_{d+1}\Big(K \otimes_R M\Big) \oplus H_d\Big(F \otimes_R M\Big).$$

Since K is a free resolution of $R/(\underline{x})$, the first summand is

$$H_{d+1}(K \otimes_R M) = \mathrm{Tor}^R_{d+1}(R/(\underline{x}), M).$$

Because K has length d, this homology vanishes:

$$H_{d+1}(K \otimes_R M) = 0.$$

On the other hand, since F is a free resolution of k, one has

$$H_d(F \otimes_R M) = \mathrm{Tor}^R_d(k, M).$$

Therefore

$$H_{d+1}\Big(\mathrm{Cone}(\sigma) \otimes_R M\Big) \cong \mathrm{Tor}^R_d(k, M).$$

Since

$$\mathrm{Cone}(\sigma) \simeq_R R/(\underline{x}, S),$$

we get

$$H_{d+1}\Big(\mathrm{Cone}(\sigma) \otimes_R M\Big) = \mathrm{Tor}^R_{d+1}(R/(\underline{x}, S), M).$$

Thus

$$\mathrm{Tor}^R_{d+1}(R/(\underline{x}, S), M) \cong \mathrm{Tor}^R_d(k, M).$$

The Contradiction Argument

Suppose M has finite length. Then

$$\mathrm{depth}_R(M) = 0.$$

If, in addition, $\mathrm{pdim}_R(M) < \infty$ and R is Cohen–Macaulay of dimension d, then the Auslander–Buchsbaum formula gives

$$\mathrm{pdim}_R(M) + \mathrm{depth}_R(M) = \mathrm{depth}\, R = d.$$

Hence

$$\mathrm{pdim}_R(M) = d.$$

Therefore

$$\mathrm{Tor}_d^R(k, M) \neq 0.$$

Combining this with the previous computation gives

$$\mathrm{Tor}_{d+1}^R(R/(\underline{x}, S), M) \cong \mathrm{Tor}_d^R(k, M) \neq 0.$$

But if $\mathrm{pdim}_R(M) = d$, then

$$\mathrm{Tor}_i^R(-, M) = 0 \quad \text{for all } i > d.$$

In particular,

$$\mathrm{Tor}_{d+1}^R(R/(\underline{x}, S), M) = 0,$$

a contradiction.

Thus the assumption

$$U_R(M) \leq n$$

cannot hold when $M \neq 0$, $\ell_R(M) < \infty$, and $\mathrm{pdim}_R(M) < \infty$. Therefore

$$U_R(M) \geq n + 1.$$

Since

$$n + 1 = U_R(R/(\underline{x})),$$

we obtain the desired lower bound

$$U_R(M) \geq U_R(R/(\underline{x})).$$

Remark 4.21. The logical structure is as follows. If M were killed by $\mathfrak{m}^n$, then the high-order chain map σ would become zero after tensoring with M. The mapping cone would split, forcing a nonzero $\mathrm{Tor}_d^R(k, M)$ to reappear as a Tor_{d+1}-group. But M has projective dimension d, so all $\mathrm{Tor}_i^R(-, M)$ for $i > d$ must vanish. This contradiction proves that M cannot be killed by $\mathfrak{m}^n$.

Proof Sketch of the Nawaj–Pollitz Bound

Assume

$$\operatorname{pdim}_R M < \infty \quad \text{and} \quad U_R(M) < \infty.$$

Let $\underline{x} = x_1, \ldots, x_d$ be a sufficiently general system of parameters whose initial forms

$$x_1^*, \ldots, x_d^*$$

form a regular sequence in $\operatorname{gr}_{\mathfrak{m}}(R)$.

Let $S \in R/(\underline{x})$ be a socle element of maximal order n. Then

$$U_R(R/(\underline{x})) = n + 1.$$

The key lemma shows that the comparison map

$$\sigma : F \to K$$

from a free resolution F of k to the Koszul resolution $K = \operatorname{Kos}(\underline{x})$ satisfies

$$\sigma(F) \subseteq \mathfrak{m}^n K.$$

If $U_R(M) \le n$, then

$$\mathfrak{m}^n \subseteq \operatorname{ann}_R(M),$$

so

$$\sigma \otimes_R M = 0.$$

The mapping cone argument then gives

$$\operatorname{Tor}_{d+1}^R(R/(\underline{x}, S), M) \cong \operatorname{Tor}_d^R(k, M) \neq 0.$$

This contradicts $\operatorname{pdim}_R(M) = d$. Therefore

$$U_R(M) \ge n + 1.$$

Consequently,

$$U_R(M) \ge n + 1 = \min\{U_R(R/(\underline{y})) \mid \underline{y} \text{ is a system of parameters}\}.$$

Some New Idea

The lecture notes end with the following diagram:

$$\begin{tikzcd}
k \arrow[r, hook, "\iota"] & R/(x_1, \ldots, x_d) \\
F \arrow[u, "\simeq"'] \arrow[r, "\sigma"] & K \arrow[u, "\simeq"]
\end{tikzcd}$$

where F is a free resolution of k, $K = \mathrm{Kos}(x_1, \ldots, x_d)$, and $\sigma : F \to K$ is a lift of the inclusion

$$\iota : k \hookrightarrow R/(x_1, \ldots, x_d).$$

In homological degree d, the component of the chain map is

$$\sigma_d : F_d \longrightarrow K_d.$$

Since the Koszul complex on d elements has top term

$$K_d \cong R,$$

we may view the top component as a map

$$\sigma_d : F_d \longrightarrow R.$$

Remark 4.22. The map $\sigma_d : F_d \to K_d = R$ is the top-degree part of the comparison map from the free resolution of k to the Koszul resolution of $R/(\underline{x})$. Understanding the image of this map is a natural way to detect the order of the socle element S, because the entire comparison map is controlled by the lift of $1 \mapsto S$. The key lemma asserts that this image is contained in $\mathfrak{m}^n$, where $n = \mathrm{ord}(S)$.