URiCA 2026

Upcoming Researchers in Commutative Algebra

University of Nebraska

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Conference Notes

URiCA 2026

Continuing the mission of KUMUNUjr

Inspired by the success of the KUMUNU commutative algebra conferences (KU, MU, NU), KUMUNUjr brought together graduate students and postdocs studying commutative algebra in the midwest to share their research and expertise while building lasting collaborations and friendships.

Upcoming Researchers in Commutative Algebra (URiCA) is the next chapter for KUMUNUjr.

Contents

1 Special Lecture: New Perspectives on the Briançon–Skoda Theorem- Karl Schwede

1.1 Integral closure: the opening example

Example 1.1. Consider the convergent power-series ring

$$R = \mathbb{C}\{x, y\}$$

and the ideal

$$I = (x^2, y^2) \subseteq R.$$

Then

$$xy \notin I,$$

but, in the language of the lecture, it "feels" as though xy should belong to a natural enlargement of I.
There are two reasons for this.

(1) **Analytic evidence.** For every $(x, y) \in \mathbb{C}^2$,

$$|xy| \leq \max\{|x|^2, |y|^2\}.$$

Indeed, if $|x| \geq |y|$, then $|xy| \leq |x|^2$, while if $|y| \geq |x|$, then $|xy| \leq |y|^2$.

(2) **Algebraic evidence.** We have

$$(xy)^2 = x^2y^2 \in I^2,$$

since

$$I^2 = (x^2, y^2)(x^2, y^2) = (x^4, x^2y^2, y^4).$$

The appropriate enlargement is the *integral closure* of an ideal.

Definition 1.2 (Integral closure in the analytic local setting)**.** Let

$$J = (f_1, \ldots, f_n) \subseteq \mathbb{C}\{x_1, \ldots, x_d\}$$

and let $h \in \mathbb{C}\{x_1, \ldots, x_d\}$. The lecture writes $h \in \overline{J}$ if either—equivalently, both—of the following conditions holds.

(1) **Analytic condition.** There is a constant $C > 0$ such that

$$|h(z)| \leq C \max\{|f_1(z)|, \ldots, |f_n(z)|\}$$

for all z in some neighborhood of the origin.

(2) **Algebraic condition.** There exists an integer $r > 0$ and elements

$$a_i \in J^i \qquad (1 \le i \le r)$$

such that

$$h^r + a_1 h^{r-1} + a_2 h^{r-2} + \cdots + a_{r-1} h + a_r = 0. \tag{1.1}$$

Remark 1.3. For $I = (x^2, y^2)$ and $h = xy$, the analytic condition holds with $C = 1$. Algebraically,

$$h^2 - x^2 y^2 = 0,$$

and the constant coefficient $-x^2y^2$ lies in I^2. Hence

$$xy \in \overline{I} \setminus I.$$

This is the first instance of the general principle that integral closure detects elements that have the same asymptotic size, or the same birational order of vanishing, as the ideal.

1.2 Classical Briançon–Skoda containments

The lecture next records the classical comparison between integral closures of high powers and ordinary powers.

Theorem 1.4 (Briançon–Skoda)**.** *Let*

$$J = (f_1, \ldots, f_n) \subseteq \mathbb{C}\{x_1, \ldots, x_d\}.$$

Then for every integer $k \ge 1$,

$$\overline{J^{n+k-1}} \subseteq J^k. \tag{1.2}$$

Also we have the dimension form

$$\overline{J^{d+k-1}} \subseteq J^k \qquad (k \ge 1). \tag{1.3}$$

In particular, taking $k = 1$ in (1.2) gives

$$\overline{J^n} \subseteq J.$$

The analytic theorem was then placed in a purely algebraic regular-ring setting.

Theorem 1.5 (Lipman–Sathaye, 1981)**.** *Let R be a regular ring and let*

$$J = (f_1, \ldots, f_n) \subseteq R.$$

Then for every $k \geq 1$,

$$\overline{J^{n+k-1}} \subseteq J^k. \tag{1.4}$$

The notes also record the dimension bound

$$\overline{J^{\dim R+k-1}} \subseteq J^k \qquad (k \geq 1). \tag{1.5}$$

The next step is to permit singularities, at the cost of measuring the exponent by the dimension.

Theorem 1.6 (Lipman–Teissier, 1981)**.** *Let $(R, \mathfrak{m})$ be a local pseudo-rational ring of dimension d, and let $J \subseteq R$ be any ideal. Then for every $k \geq 1$,*

$$\overline{J^{d+k-1}} \subseteq J^k. \tag{1.6}$$

1.3 Pseudo-rational singularities and the blowup viewpoint

The lecture lists the following examples.

Example 1.7 (Examples recorded in the lecture)**.** (1) Regular rings are pseudo-rational.

(2) The quadric cone
$$\Bbbk[x, y, z]/(x^2 + y^2 - z^2)$$
is pseudo-rational when $\operatorname{char}(\Bbbk) \neq 2$.

(3) The cubic cone
$$\Bbbk[x, y, z]/(x^3 + y^3 - z^3)$$
is recorded as never pseudo-rational.

The next change of viewpoint is geometric: think of elements of R as functions on $\operatorname{Spec} R$ and modify $\operatorname{Spec} R$ so that the ideal becomes locally principal.

1.3.1 Normalized blowup and integral closure

Let $J \subseteq R$ be an ideal, and let
$$\pi\colon Y \longrightarrow \operatorname{Spec} R$$
be the normalized blowup of J in the setting where this normalization is defined. On the standard affine charts of Y, the pulled-back ideal $J\mathcal{O}_Y$ is principal (equivalently, invertible).

The opening example reappears very concretely.

Example 1.8 (The normalized blowup of (x^2, y^2))**.** Take

$$I = (x^2, y^2) \subseteq \mathbb{C}[x, y].$$

The normalized blowup is covered by the familiar charts

$$\mathbb{C}\left[\frac{x}{y}, y\right] \qquad \text{and} \qquad \mathbb{C}\left[x, \frac{y}{x}\right].$$

On the first chart,

$$\begin{aligned} I\,\mathbb{C}\left[\frac{x}{y}, y\right] &= \left(\left(\frac{x}{y}\right)^2 y^2, y^2\right) \\ &= (y^2), \end{aligned}$$

and

$$xy = \frac{x}{y}\, y^2 \in (y^2).$$

On the second chart,

$$\begin{aligned} I\,\mathbb{C}\left[x, \frac{y}{x}\right] &= \left(x^2, \left(\frac{y}{x}\right)^2 x^2\right) \\ &= (x^2), \end{aligned}$$

and

$$xy = \frac{y}{x}\, x^2 \in (x^2).$$

Thus xy belongs to the pullback of I on every chart of the normalized blowup.

We have the normalized-blowup criterion in the formula

$$\overline{J} = \Gamma(Y, J\mathcal{O}_Y) \cap R. \tag{1.7}$$

Remark 1.9. The formula says that an element $h \in R$ lies in $\overline{J}$ precisely when, after passing to the normalized blowup where J becomes invertible, the pullback of h is a local section of $J\mathcal{O}_Y$. In Example 1.8, this is exactly what happens to xy.

1.4 Derived global sections and the L-complex

We now replace ordinary global sections by derived global sections. Form

$$\mathbf{R}\Gamma(Y, \mathcal{O}_Y),$$

a complex of R-modules whose cohomology is the sheaf cohomology:

$$H^i\Big(\mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big) \cong H^i(Y, \mathcal{O}_Y). \tag{1.8}$$

Let

$$J = (f_1, \ldots, f_n), \qquad \underline{f} = (f_1, \ldots, f_n).$$

Instead of working only with the quotient R/J, we can replace it by the Koszul complex:

$$H_0\Big(\mathrm{Kos}(\underline{f})\Big) \cong R/J.$$

For higher powers, the analogous replacement is a complex denoted

$$L^k(\underline{f}),$$

with

$$H_0\Big(L^k(\underline{f})\Big) \cong R/J^k. \tag{1.9}$$

Remark 1.10. $L^k(\underline{f})$ is the Buchsbaum–Eisenbud L-complex associated to the kth power of the ideal generated by $\underline{f}$. For $k = 1$ it is the Koszul complex. If $f_1, \ldots, f_n$ form a regular sequence, $L^k(\underline{f})$ is a free resolution of R/J^k; in general, the feature used in the lecture is the zeroth-homology identity (1.9).

1.4.1 The derived vanishing on the blowup

Write the pulled-back ideal as

$$J\mathcal{O}_Y = \mathcal{O}_Y(-E),$$

where E denotes the effective exceptional divisor corresponding to J on the blowup.

Lemma 1.11 (The key derived vanishing). *For every $k \geq 1$, the canonical composite*

$$\mathcal{O}_Y\Big(-(n+k-1)E\Big) \longrightarrow \mathcal{O}_Y \longrightarrow L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathcal{O}_Y \tag{1.10}$$

is zero in the derived category $\mathrm{D}(Y)$.

Remark 1.12. The sheaf $\mathcal{O}_Y(-(n+k-1)E)$ is the $(n+k-1)$st power of the invertible ideal $J\mathcal{O}_Y$. Thus Lemma 1.11 says that, on the space where J has been principalized, sufficiently high order along E kills the map into the complex that detects R/J^k.

Applying derived global sections to (1.10) gives the global form of the same vanishing.

Theorem 1.13 (Global derived Briançon–Skoda mechanism)**.** *The canonical map*

$$\mathbf{R}\Gamma\Big(Y, \mathcal{O}_Y\Big(-(n+k-1)E\Big)\Big) \longrightarrow L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \tag{1.11}$$

is zero in $\mathrm{D}(R)$*. Consequently, elements of* $\overline{J^{n+k-1}}$ *map to zero in*

$$H_0\Big(L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big).$$

Proof. The first assertion is obtained by applying the derived functor $\mathbf{R}\Gamma(Y, -)$ to Lemma 1.11 and using the projection formula. For the second assertion, combine the normalized-blowup description of integral closure with the identification

$$J\mathcal{O}_Y = \mathcal{O}_Y(-E).$$

Thus an element of $\overline{J^{n+k-1}}$ gives a global section of the source of (1.11), hence has zero image in zeroth homology of the target. □

Remark 1.14. At this point, the argument has not yet shown that an element of $\overline{J^{n+k-1}}$ lies in J^k. It has shown that the element dies after mapping into a derived object built from the blowup. The remaining issue is to compare that derived object back to R/J^k. This is exactly where a splitting of

$$R \longrightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$$

enters.

1.5 Pseudo-rationality as a splitting mechanism

Definition 1.15. In the setting under discussion, we write R is pseudo-rational precisely when R is Cohen–Macaulay and, for every relevant blowup

$$\pi\colon Y \longrightarrow \operatorname{Spec} R,$$

the natural map

$$R \longrightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \tag{1.12}$$

splits in the derived category. Thus there is a morphism

$$\varphi \colon \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \longrightarrow R$$

such that the composite

$$R \longrightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\varphi} R$$

is the identity.

The splitting turns the derived vanishing into actual ideal membership.

Theorem 1.16 (Full n-generated Briançon–Skoda from splitting)**.** *Let $J = (f_1, \ldots, f_n) \subseteq R$, and let $Y \to \operatorname{Spec} R$ be the normalized blowup appearing above. Assume that*

$$R \longrightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$$

admits a retraction in $\mathrm{D}(R)$. Then, for every $k \geq 1$,

$$\overline{J^{n+k-1}} \subseteq J^k. \tag{1.13}$$

In particular, the lecture applies this to the pseudo-rational situation.

Proof. Let

$$\varphi \colon \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \longrightarrow R$$

be a retraction. Tensoring with the perfect complex $L^k(\underline{f})$ gives a map

$$L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \longrightarrow L^k(\underline{f}) \otimes^{\mathbf{L}}_R R.$$

Taking zeroth homology and using (1.9) yields a comparison with R/J^k. Schematically,

$$\begin{tikzcd} R \arrow[r] \arrow[dr, two heads, "\mathrm{mod}\ J^k"'] & H_0\Big(L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big) \arrow[d, "H_0(1 \otimes \varphi)"] \\ & H_0\Big(L^k(\underline{f}) \otimes^{\mathbf{L}}_R R\Big) \cong R/J^k. \end{tikzcd}$$

By Theorem 1.13, every element of $\overline{J^{n+k-1}}$ maps to zero along the top arrow. Therefore its image in R/J^k is also zero, which is exactly (1.13). □

Remark 1.17. For an n-generated ideal in a d-dimensional pseudo-rational ring, the classical statement in Theorem 1.6 uses the exponent $d+k-1$. The derived splitting argument recovers the generator-sensitive exponent $n+k-1$. When $n<d$, this is a genuinely stronger containment.

1.6 Alterations and the uniform Briançon–Skoda direction

Now how one can retain a useful remnant of the splitting argument even when R itself need not satisfy the pseudo-rational splitting property?

Suppose

$$\eta\colon R \longrightarrow \mathbf{R}\Gamma(Z, \mathcal{O}_Z)$$

comes from a map

$$Z \longrightarrow \operatorname{Spec} R$$

which may be one of the following objects listed in the lecture:

- a resolution of singularities;
- a regular alteration;
- a truncated regular alteration hypercover.

In derived global sections one has a factorization of the form

$$\begin{tikzcd} & \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \arrow[dr] & \\ R \arrow[ur] \arrow[rr, "\eta"'] & & \mathbf{R}\Gamma(Z, \mathcal{O}_Z), \end{tikzcd}$$

for the relevant blowups $Y \to \operatorname{Spec} R$.

Let

$$\phi\colon \mathbf{R}\Gamma(Z, \mathcal{O}_Z) \longrightarrow R \tag{1.14}$$

such that

$$\phi \circ \eta = c \cdot \mathrm{id}_R \tag{1.15}$$

for some nonzero element $c \in R$.

Remark 1.18. For a pseudo-rational ring one has $c=1$: the map back is an actual retraction. In the alteration setting one may only be able to split after passing to the generic point. Clearing denominators then produces a nonzero c for which the composite is multiplication by c. Thus c measures the failure of an honest splitting over all of R.

Proposition 1.19 (A uniform multiplier). *Let $J = (f_1, \ldots, f_n) \subseteq R$. Under the factorization and map* (1.14)–(1.15) *described above, the lecture obtains*

$$c\,\overline{J^{n+k-1}} \subseteq J^k \qquad (k \geq 1). \tag{1.16}$$

The important point is that c is chosen from the ambient birational/alteration construction rather than separately from each ideal J.

Proof mechanism. Take $h \in \overline{J^{n+k-1}}$. By Theorem 1.13, its image in

$$H_0\Big(L^k(\underline{f}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big)$$

is zero. Pass through the factorization to $\mathbf{R}\Gamma(Z, \mathcal{O}_Z)$, then apply $1 \otimes \phi$. Because the composite $R \to \mathbf{R}\Gamma(Z, \mathcal{O}_Z) \to R$ is multiplication by c, the image of h in R/J^k becomes the class of ch. Hence $ch = 0$ in R/J^k, or equivalently

$$ch \in J^k.$$

This proves (1.16). □

Corollary 1.20 (Uniform Briançon–Skoda; lecture formulation). *For an excellent, reduced ring R of finite dimension, there exists an integer $\ell > 0$ such that for every ideal $J \subseteq R$ and every $k \geq 1$,*

$$\overline{J^{k+\ell}} \subseteq J^k. \tag{1.17}$$