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Commutative Algebra

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Conference Notes

RayFest

A conference in celebration of the mathematics of Raymond C. Heitmann

Contents

1 Lecture 1 – Finding the Nilradiacal by Craig Huenke

The Basic Problem

Let

$$S = K[x_1, \dots, x_n] \qquad \text{or} \qquad S = K[[x_1, \dots, x_n]]$$

and let $I \subseteq S$ be an ideal.

Question 1.1. How can one effectively find the radical $\sqrt{I}$?

The usual descriptions are

$$\sqrt{I} = \bigcap_{\substack{P \supseteq I \\ P \text{ prime}}} P$$

and

$$\sqrt{I} = \{f \in S \mid f^N \in I \text{ for some } N \geq 1\}.$$

Equivalently, one may characterize membership in $\sqrt{I}$ by testing all field-valued points annihilating I. Namely,

$$\sqrt{I} = \left\{ f \in S \;\middle|\; \begin{array}{l} \text{for every homomorphism } Q : S \to L \text{ to a field } L \\ \text{with } Q(I) = 0, \text{ one also has } Q(f) = 0 \end{array} \right\}.$$

Remark 1.2. The last characterization follows because a homomorphism $Q : S \to L$ to a field has prime kernel. Thus $Q(I) = 0$ means $I \subseteq \ker Q$, and $Q(f) = 0$ means $f \in \ker Q$. Varying over all such maps is equivalent to varying over all prime ideals $P \supseteq I$, since each quotient S/P embeds into its fraction field.

A Matrix Nilpotence Example

Example 1.3. Let

$$S = K[X_{ij}]_{1 \leq i,j \leq n}$$

be the polynomial ring in the entries of a generic $n \times n$ matrix $X = (X_{ij})$, and let $I = I_1(X^n)$, the ideal generated by all entries of X^n.
We ask: what is $\sqrt{I}$?
If $\overline{X}$ denotes the image of X modulo a prime ideal $P \supseteq I$, then $\overline{X}^n = 0$, so $\overline{X}$ is nilpotent. Hence its characteristic polynomial is T^n. Therefore the coefficients of the characteristic polynomial vanish:

$$c_1 = X_{11} + \cdots + X_{nn}, \qquad c_2, \dots, \qquad c_n = \det(X).$$

Consequently,

$$(c_1, \dots, c_n) \subseteq \sqrt{I}.$$

Conversely, the vanishing of all coefficients $(c_1, \dots, c_n)$ forces the characteristic polynomial of $\overline{X}$ to be T^n, and by Cayley–Hamilton one has $\overline{X}^n = 0$. Thus

$$\sqrt{I} = \sqrt{(c_1, \dots, c_n)}.$$

A natural further question is whether

$$\sqrt{I} = (c_1, \dots, c_n).$$

Remark 1.4. The implication from $c_1 = \cdots = c_n = 0$ to $X^n = 0$ uses the Cayley–Hamilton theorem: if the characteristic polynomial of X is T^n, then $X^n = 0$ in the quotient ring.

Testing Membership in the Radical

A fundamental computational test for membership in the radical is the Rabinowitsch trick.

Lemma 1.5 (Rabinowitsch Trick)**.** *Let S be a ring, let $I \subseteq S$ be an ideal, and let $f \in S$. Then*

$$f \in \sqrt{I} \quad \iff \quad 1 \in (IS[t], 1 - ft) \subseteq S[t].$$

Proof. Suppose $f \in \sqrt{I}$. Then $f^N \in I$ for some $N \geq 1$. In $S[t]$, we have

$$1 - (ft)^N = (1 - ft)(1 + ft + \cdots + (ft)^{N-1}).$$

Since $f^N \in I$, the term $(ft)^N = f^N t^N$ lies in $IS[t]$. Hence

$$1 \in (IS[t], 1 - ft).$$

Conversely, suppose

$$1 \in (IS[t], 1 - ft).$$

Modulo I, this says

$$1 \in ((S/I)[t])(1 - \overline{f}t).$$

Thus $1 - \overline{f}t$ is a unit in $(S/I)[t]$. A polynomial of the form $1 - at$ is a unit in $R[t]$ only if a is nilpotent. Hence $\overline{f}$ is nilpotent in S/I, which means $f \in \sqrt{I}$. $\square$

Moreover,

$$S_f \cong \frac{S[t]}{(1-ft)}.$$

Thus S_f is an S-algebra essentially of finite type.

Effective Exponents

Theorem 1.6 (Koll'ar). *Let*

$$I = (f_1, \ldots, f_k) \subseteq S$$

where $S = K[x_1, \ldots, x_n]$, *and suppose*

$$\deg f_1 = d_1 \geq d_2 \geq \cdots \geq d_k > 2.$$

Then:

(1) *if* $k \geq n$, *then*

$$(\sqrt{I})^{d_1 \cdots d_{n-1} d_k} \subseteq I;$$

(2) *if* $k < n$, *then*

$$(\sqrt{I})^{d_1 \cdots d_k} \subseteq I.$$

Remark 1.7. This theorem gives an effective uniform exponent N such that $(\sqrt{I})^N \subseteq I$. Hence, in principle, radical membership can be checked by verifying whether sufficiently high powers of a given element lie in I.

A Regularity Question of Rothschild–Baumend Type

Let

$$R = \mathbb{C}[[f_1, \ldots, f_n]] \subseteq \mathbb{C}[[x_1, \ldots, x_n]] = S.$$

Let $P \in \operatorname{Spec}(R)$. Assume that

$$S/\sqrt{PS}$$

is regular. One may ask whether

$$R/P$$

is regular.

The notes also record the special situation

$$x_i^d R = P, \qquad d \geq 2.$$

Remark 1.8. This is a question about whether regularity descends from the radical extension in S back to the subring R. The notation $x_i^d R = P$ suggests a situation where the prime P is controlled by powers of the ambient parameters.

A Baby Example: One Variable

Example 1.9 (Baby Example)**.** Let

$$f \in S = K[x], \qquad \operatorname{char}(K) = 0.$$

We want to find

$$\sqrt{(f)}.$$

Since S is a unique factorization domain, write

$$f = f_1^{a_1} \cdots f_d^{a_d},$$

where the f_i are pairwise relatively prime. Then

$$\sqrt{(f)} = (f_1 \cdots f_d).$$

Now consider

$$f' = f_1^{a_1-1} \cdots f_d^{a_d-1} g, \qquad (g, f) = 1.$$

Then

$$(f) : f' = \left(f : f_1^{a_1-1} \cdots f_d^{a_d-1}\right) = (f_1 \cdots f_d) = \sqrt{(f)}.$$

Remark 1.10. The element f' removes precisely the repeated factors from f. Taking the colon ideal $(f) : f'$ leaves the squarefree part of f, which generates the radical of (f).

A Higher-Dimensional Complete Intersection Example

Example 1.11 (Generalized Example)**.** Let

$$I = (f_1, \ldots, f_n) \subseteq K[x_1, \ldots, x_n], \qquad \operatorname{char}(K) = 0.$$

Suppose

$$\sqrt{I} = (x_1, \ldots, x_n).$$

How can one compute
$$\sqrt{(f_1, \ldots, f_n)}?$$
Ideally, one wants to find the socle of S/I. If g generates the socle, then
$$(I : g) = (x_1, \ldots, x_n) = \sqrt{I}.$$
The Scheja–Storch theorem says that
$$g = \det\left(\frac{\partial f_i}{\partial x_j}\right)$$
is such a socle generator, in the complete intersection case.

Remark 1.12. If $I = (f_1, \ldots, f_n)$ is an $\mathfrak{m}$-primary complete intersection in $S = K[x_1, \ldots, x_n]$, then S/I is Artinian Gorenstein. Hence its socle is one-dimensional over K, and the Jacobian determinant generates the socle.

A Warning

The preceding Jacobian determinant method fails if the generators are not a regular sequence. Take
$$I = (x^2, xy, y^2) \subseteq K[x, y].$$
The Jacobian matrix of the three generators (x^2, xy, y^2) with respect to (x, y) is
$$\left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix}.$$
Taking the 2×2 minors, assuming $\mathrm{char}(K) = 0$, gives
$$I_2\left(\frac{\partial f_i}{\partial x_j}\right) = (x, y)^2 = I.$$
Thus the Jacobian minors do not recover $\sqrt{I} = (x, y)$ in this non-complete-intersection example.

The Eisenbud–Huneke–Vasconcelos Formula

Let
$$S = K[x_1, \ldots, x_n], \qquad I = (f_1, \ldots, f_k) \subseteq S,$$
and let
$$d = \dim S/I.$$

The Jacobian matrix is

$$\operatorname{Jac}(I) = \left(\frac{\partial f_i}{\partial x_j}\right).$$

Theorem 1.13 (Eisenbud–Huneke–Vasconcelos)**.** *Assume that I is unmixed, meaning that all minimal primes of I have the same height, and assume that I is generically a complete intersection, meaning that*

$$I_P$$

can be generated by $\operatorname{ht} P$ *elements for every*

$$P \in \operatorname{Min}(S/I).$$

Then

$$\sqrt{I} = I : I_{n-d}\left(\frac{\partial f_i}{\partial x_j}\right).$$

Remark 1.14. Since $d = \dim S/I$, one has $n - d = \operatorname{ht} I$ when I is equidimensional. Thus the relevant minors are the minors of size equal to the codimension of I.

In general, three remaining problems arise:

(1) Can one reduce to I equidimensional?

(2) Can one reduce to I having no embedded primes?

(3) Can one reduce to I generically a complete intersection?

Problems (1) and (2) are not too difficult.

Removing Components of the Wrong Codimension

Example 1.15. Suppose $I \subseteq S$ has codimension c, and let J be the intersection of the primary components of I of codimension c. Then

$$J = \operatorname{ann}\Big(\operatorname{Ext}^c_S(S/I, S)\Big).$$

Equivalently, J is the unmixed part of I in codimension c.

Hint. Let $\underline{x} = x_1, \dots, x_c$ be a maximal S-regular sequence contained in I. Then show

$$J = (\underline{x}) : \Big((\underline{x}) : I\Big) = \operatorname{ann}\Big(\operatorname{Ext}^c_S(S/I, S)\Big).$$

Remark 1.16. The equality

$$(\underline{x}) : \big((\underline{x}) : I\big)$$

is a standard linkage-theoretic description of the unmixed part of I when $\underline{x}$ is a maximal regular sequence contained in I. The Ext module detects the codimension-c part of S/I.

Rather than finding an ideal J such that

$$\sqrt{I} = I : J,$$

it is enough, for an iterative algorithm, to find an ideal J such that

$$I \subsetneq I : J \subseteq \sqrt{I}.$$

Indeed, replacing I by $I : J$ strictly enlarges the ideal while staying inside the radical.

A Test Family

Example 1.17 (Test Example)**.** Let

$$f_{mn} = x_1^n + \cdots + x_m^n$$

and define

$$I_{mn} = (f_{m2}, f_{m3}, \ldots, f_{mn}).$$

The problem is to find

$$\sqrt{I_{mn}}.$$

A Criterion of Vasconcelos

Let

$$S^b \xrightarrow{\phi} S^n \longrightarrow I \longrightarrow 0$$

be a presentation of the ideal I.

Theorem 1.18 (Vasconcelos)**.** *With notation as above,*

$$I_1(\phi) \subseteq I \quad \Longleftrightarrow \quad I \text{ is generated by a regular sequence.}$$

Here $I_1(\phi)$ denotes the ideal generated by the entries, equivalently the 1×1 minors, of ϕ.

Proof. If I is generated by a regular sequence, then the presentation of I is given by the

Koszul syzygies. The entries appearing in the Koszul syzygy matrix lie in I, so

$$I_1(\phi) \subseteq I.$$

Conversely, suppose

$$I_1(\phi) \subseteq I.$$

Tensoring the presentation

$$S^b \xrightarrow{\phi} S^n \longrightarrow I \longrightarrow 0$$

with S/I gives

$$(S/I)^b \xrightarrow{0} (S/I)^n \longrightarrow I/I^2 \longrightarrow 0.$$

The map is zero because all entries of ϕ lie in I. Hence

$$I/I^2$$

is a free S/I-module. By a theorem of Vasconcelos, this implies that I is generated by a regular sequence. □

Remark 1.19. The last implication uses the fact that, under the relevant hypotheses, freeness of the conormal module I/I^2 forces I to be locally generated by a regular sequence. This is one of the central mechanisms behind detecting complete intersections from presentations.

Detecting the Number of Generators Locally

Let

$$S^b \xrightarrow{\phi} S^n \longrightarrow I \longrightarrow 0$$

be a presentation. The notes record the following observation:

$$\mu(I_P) \leq k \quad \iff \quad I_{n-k}(\phi) \not\subseteq P.$$

Remark 1.20. Here $\mu(I_P)$ denotes the minimal number of generators of the localized ideal I_P. The condition that I_P can be generated by at most k elements is equivalent to saying that, after localizing at P, the presentation matrix has enough rank to eliminate $n-k$ generators. This is detected by the nonvanishing of an appropriate Fitting ideal.

If I is unmixed of codimension c, then I is generated by a complete intersection if and only if

$$\mathrm{ht}\Big(I_{n-c}(\phi)\Big) \geq c+1.$$

A Colon Step Toward the Radical

Theorem 1.21. *Let*

$$S = K[x_1, \dots, x_n], \qquad I \subseteq S.$$

Let

$$S^b \xrightarrow{\phi} S^n \longrightarrow I \longrightarrow 0$$

be a presentation of I, and let

$$c = \mathrm{ht}(I).$$

Assume that I is not generated by a complete intersection. Choose k maximal such that

$$J := I_{n-k}(\phi)$$

has height c. Then

$$I \subsetneq I : J \subseteq \sqrt{I}.$$

Proof. The containment

$$I \subseteq I : J$$

is automatic. The point is that J detects the locus where I fails to have sufficiently few generators. By the choice of k, the ideal J has the same height as I, and the maximality condition forces J to meet the relevant minimal components of I.

For a minimal prime $P \in \mathrm{Min}(S/I)$, the localization I_P is controlled by the localized presentation matrix ϕ_P. The Fitting ideal J was chosen so that, on the generic points of $V(I)$, the local number of generators is constrained. The failure of I to be a complete intersection produces an element of J that is not already absorbed by I, so the colon ideal $I : J$ strictly contains I.

On the other hand, at every prime $P \supseteq I$, the construction ensures that no element outside P can enter $I : J$. Hence

$$I : J \subseteq \bigcap_{P \supseteq I} P = \sqrt{I}.$$

Therefore

$$I \subsetneq I : J \subseteq \sqrt{I}.$$

□

Remark 1.22. The theorem supplies an iterative method for enlarging I inside its radical. Once one finds such a J, replacing I by $I : J$ moves closer to $\sqrt{I}$. Repeating this step can be viewed as an algorithmic strategy for computing radicals.

Algorithmic Summary

The lecture ends with the following algorithmic principle:

Repeat the colon enlargement step.

More explicitly, starting from I, one seeks ideals J such that

$$I \subsetneq I : J \subseteq \sqrt{I}.$$

Then replace I by $I : J$, and repeat until no further enlargement occurs. In favorable situations, this process terminates at $\sqrt{I}$.

2 Lecture 2 —Extremal Ideals & Properties of Square-free Monomial Ideals by Susan Morey

Background and Definitions

Let K be a field and let

$$R = K[x_1, \ldots, x_n].$$

Let $a \in \mathbb{N}$, and let

$$I = (m_1, \ldots, m_a) \subseteq R$$

be a square-free monomial ideal; that is, each m_i is a square-free monomial in the variables $x_1, \ldots, x_n$. Denote

$$[n] = \{1, 2, \ldots, n\}.$$

Similarly, for $a \in \mathbb{N}$, write

$$[a] = \{1, 2, \ldots, a\}.$$

Definition 2.1. Given $a \in \mathbb{N}$, define the polynomial ring

$$S_{[a]} = K\Big[Y_A \mid \emptyset \neq A \subseteq [a]\Big].$$

Thus $S_{[a]}$ has one variable Y_A for every nonempty subset $A \subseteq [a]$.

Remark 2.2. For a set of size a, there are $2^a - 1$ nonempty subsets. Hence $S_{[a]}$ is a polynomial ring over K in $2^a - 1$ variables.

Example 2.3. For $a = 3$,

$$S_{[3]} = K[Y_{\{1\}}, Y_{\{2\}}, Y_{\{3\}}, Y_{\{1,2\}}, Y_{\{1,3\}}, Y_{\{2,3\}}, Y_{\{1,2,3\}}].$$

For typographical convenience, we often write Y_{12} for $Y_{\{1,2\}}$, Y_{123} for $Y_{\{1,2,3\}}$, and so on.

Definition 2.4. For each $i \in [a]$, define

$$\varepsilon_i = \prod_{\substack{i \in A \\ \emptyset \neq A \subseteq [a]}} Y_A \in S_{[a]}.$$

The *extremal ideal* on a generators is

$$\mathcal{E}_a = (\varepsilon_1, \ldots, \varepsilon_a) \subseteq S_{[a]}.$$

Example 2.5. For $a = 3$ and $i = 2$, the nonempty subsets of $[3]$ containing 2 are

$$\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}.$$

Therefore

$$\varepsilon_2 = Y_{\{2\}} Y_{\{1,2\}} Y_{\{2,3\}} Y_{\{1,2,3\}} = Y_2 Y_{12} Y_{23} Y_{123}.$$

Since $\mathcal{E}_a = (\varepsilon_1, \ldots, \varepsilon_a)$, its r-th power is generated by all monomials of the form

$$\varepsilon_1^{r_1} \cdots \varepsilon_a^{r_a}, \qquad r_1 + \cdots + r_a = r, \qquad r_i \in \mathbb{N}.$$

One is often interested in the quotient ring $S_{[a]}/\mathcal{E}_a$, as well as in the powers $\mathcal{E}_a^r$.

Definition 2.6. For each $k \in [n]$, define

$$A_k = \{j \in [a] \mid x_k \text{ divides } m_j\}.$$

Thus A_k records exactly which of the generators $m_1, \ldots, m_a$ are divisible by x_k.

Example 2.7. Let

$$I = (x_1x_2x_5,\ x_2x_5x_6,\ x_1x_3x_4x_6) \subseteq K[x_1, \ldots, x_6].$$

Here $a = 3$, and we set

$$m_1 = x_1x_2x_5, \qquad m_2 = x_2x_5x_6, \qquad m_3 = x_1x_3x_4x_6.$$

Then

$$\begin{aligned} A_1 &= \{1,3\}, \\ A_2 &= \{1,2\} = A_5, \\ A_3 &= \{3\} = A_4, \\ A_6 &= \{2,3\}. \end{aligned}$$

Definition 2.8. Associated to $I = (m_1, \ldots, m_a)$, define a K-algebra homomorphism

$$\Psi_I : S_{[a]} \longrightarrow R$$

on the variables of $S_{[a]}$ by

$$\Psi_I(Y_A) = \begin{cases} \prod_{k \mid A_k = A} x_k, & \text{if there exists } k \text{ such that } A_k = A, \\ 1, & \text{otherwise.} \end{cases}$$

Remark 2.9. The map Ψ_I sends the auxiliary variable Y_A to the product of all original variables x_k that divide exactly the generators indexed by A. If no variable x_k has this incidence pattern, then Y_A is sent to 1.

Example 2.10. Continuing with

$$I = (x_1x_2x_5,\ x_2x_5x_6,\ x_1x_3x_4x_6),$$

we have

$$\begin{aligned} \Psi_I(Y_{\{1,3\}}) &= x_1, \\ \Psi_I(Y_{\{1,2\}}) &= x_2x_5, \\ \Psi_I(Y_{\{3\}}) &= x_3x_4, \\ \Psi_I(Y_{\{2,3\}}) &= x_6. \end{aligned}$$

For the subsets $A \subseteq [3]$ that do not occur as some A_k, the image is 1. Thus

$$\Psi_I(Y_{\{1\}}) = \Psi_I(Y_{\{2\}}) = \Psi_I(Y_{\{1,2,3\}}) = 1.$$

Fact. The monomials $\Psi_I(Y_A)$, as A ranges over the nonempty subsets of $[a]$, are square-free monomials or 1, and their supports are pairwise disjoint.

Proof. Each variable x_k belongs to exactly one image, namely to $\Psi_I(Y_{A_k})$. Therefore no variable x_k can occur in both $\Psi_I(Y_A)$ and $\Psi_I(Y_B)$ when $A \neq B$. Since the original ideal I is square-free, each x_k occurs at most once in each generator m_j, and hence the images $\Psi_I(Y_A)$ are square-free. □

Fact. For every $i \in [a]$,

$$\Psi_I(\varepsilon_i) = m_i.$$

Consequently,

$$\Psi_I(\mathcal{E}_a) = I.$$

Proof. By definition,

$$\varepsilon_i = \prod_{\substack{i \in A \\ \emptyset \neq A \subseteq [a]}} Y_A.$$

Applying Ψ_I, we obtain

$$\Psi_I(\varepsilon_i) = \prod_{\substack{i \in A \\ \emptyset \neq A \subseteq [a]}} \Psi_I(Y_A).$$

A variable x_k appears in this product precisely when $A_k = A$ for some A containing i. This is equivalent to $i \in A_k$, which is equivalent to $x_k \mid m_i$. Therefore the product is exactly m_i. □

Example 2.11. Using the earlier computation

$$\varepsilon_2 = Y_2 Y_{12} Y_{23} Y_{123},$$

we have

$$\begin{aligned}
\Psi_I(\varepsilon_2) &= \Psi_I(Y_2 Y_{12} Y_{23} Y_{123}) \\
&= \Psi_I(Y_2)\Psi_I(Y_{12})\Psi_I(Y_{23})\Psi_I(Y_{123}) \\
&= 1 \cdot (x_2 x_5) \cdot x_6 \cdot 1 \\
&= x_2 x_5 x_6 \\
&= m_2.
\end{aligned}$$

Fact. For every $r \geq 1$,

$$\Psi_I(\mathcal{E}_a^r) = I^r.$$

Proof. The ideal $\mathcal{E}_a^r$ is generated by all monomials

$$\varepsilon_1^{r_1} \cdots \varepsilon_a^{r_a}, \qquad r_1 + \cdots + r_a = r.$$

Applying Ψ_I, these generators map to

$$m_1^{r_1} \cdots m_a^{r_a}, \qquad r_1 + \cdots + r_a = r,$$

which are precisely the standard monomial generators of I^r. □

Fact. For every $i \geq 0$ and $r \geq 1$, the total Betti numbers satisfy

$$\beta_i(I^r) \leq \beta_i(\mathcal{E}_a^r).$$

Remark 2.12. The extremal ideal $\mathcal{E}_a$ is universal for square-free monomial ideals with a generators. The map Ψ_I specializes the variables of $S_{[a]}$ by replacing each auxiliary variable Y_A with a square-free monomial, or with 1. Since the supports of these images are disjoint, this specialization preserves least common multiples in a controlled way. Consequently, homological invariants of I^r are bounded above by the corresponding invariants of $\mathcal{E}_a^r$.

Primes and Intersections

Let P be a monomial prime ideal in $S_{[a]}$. Then

$$P = (Y_{B_1}, \dots, Y_{B_t})$$

for some nonempty subsets $B_1, \dots, B_t \subseteq [a]$. Applying Ψ_I, one obtains

$$\Psi_I(P) = \Big(\Psi_I(Y_{B_1}), \dots, \Psi_I(Y_{B_t})\Big).$$

More explicitly,

$$\Psi_I(P) = \left(\prod_{k \mid A_k = B_1} x_k, \dots, \prod_{k \mid A_k = B_t} x_k\right).$$

This ideal can be expressed as an intersection of monomial prime ideals in R. Namely, if

$$\Psi_I(Y_{B_j}) = \prod_{k \in C_j} x_k,$$

where

$$C_j = \{k \in [n] \mid A_k = B_j\},$$

then

$$\Psi_I(P) = \Big(\prod_{k \in C_1} x_k, \dots, \prod_{k \in C_t} x_k\Big) = \bigcap_{(k_1,\dots,k_t) \in C_1 \times \cdots \times C_t} (x_{k_1}, \dots, x_{k_t}),$$

with the convention that empty products are 1, and if one of the sets C_j is empty then $\Psi_I(Y_{B_j}) = 1$, hence $\Psi_I(P) = R$.

Remark 2.13. When $\Psi_I(P) \neq R$, the primes appearing in the above intersection have height t, which is the height of

$$P = (Y_{B_1}, \dots, Y_{B_t}).$$

Fact. If $P = (Y_{B_1}, \dots, Y_{B_t})$, then

$$\Psi_I(P^r) = \bigcap_{(k_1,\dots,k_t)\in C_1\times\cdots\times C_t} (x_{k_1}, \dots, x_{k_t})^r,$$

again with the convention that if $\Psi_I(P) = R$, then $\Psi_I(P^r) = R$.

Fact. For monomials $M, N \in S_{[a]}$,

$$\Psi_I(\operatorname{lcm}(M, N)) = \operatorname{lcm}(\Psi_I(M), \Psi_I(N)).$$

Proof. Since the monomials $\Psi_I(Y_A)$ have pairwise disjoint support, taking least common multiples commutes with replacing each Y_A by $\Psi_I(Y_A)$. Thus the exponent of any variable x_k on both sides is determined by the exponent of Y_{A_k} in the corresponding least common multiple. □

Fact. For monomial ideals $J, K \subseteq S_{[a]}$,

$$\Psi_I(J \cap K) = \Psi_I(J) \cap \Psi_I(K).$$

Proof. Intersections of monomial ideals are controlled by least common multiples of monomial generators. If $J = (u_1, \dots, u_p)$ and $K = (v_1, \dots, v_q)$, then

$$J \cap K = \big(\operatorname{lcm}(u_i, v_j) \mid 1 \le i \le p,\ 1 \le j \le q\big).$$

Applying Ψ_I and using the previous fact gives

$$\begin{aligned}\Psi_I(J \cap K) &= \big(\Psi_I(\operatorname{lcm}(u_i, v_j)) \mid i, j\big)\\ &= \big(\operatorname{lcm}(\Psi_I(u_i), \Psi_I(v_j)) \mid i, j\big)\\ &= \Psi_I(J) \cap \Psi_I(K).\end{aligned}$$

□

The map Ψ_I sends elements of

$$\operatorname{Ass}(S_{[a]}/\mathcal{E}_a^r)$$

either to R or to elements of

$$\operatorname{Ass}(R/I^r).$$

Equivalently, for $P \in \operatorname{Ass}(S_{[a]}/\mathcal{E}_a^r)$, one has

$$\Psi_I(P) = R \qquad \text{or} \qquad \Psi_I(P) \in \operatorname{Ass}(R/I^r).$$

Remark 2.14. For monomial ideals, associated primes can be detected by colon ideals with respect to monomials. The compatibility of Ψ_I with least common multiples and intersections ensures that colon computations for $\mathcal{E}_a^r$ specialize to colon computations for I^r, except in the case where some relevant prime is sent to the unit ideal R.

Applications

Application 2.15 (Persistence)**.** *Suppose that $\mathcal{E}_a$ satisfies persistence; that is,*

$$P \in \operatorname{Ass}(S_{[a]}/\mathcal{E}_a^r) \implies P \in \operatorname{Ass}(S_{[a]}/\mathcal{E}_a^{r+1})$$

for all $r \geq 1$. If $\Psi_I(P) \neq R$, then $\Psi_I(P)$ occurs in the associated primes of both R/I^r and R/I^{r+1}. Thus persistence for the extremal ideal controls persistence phenomena for the specialized square-free monomial ideal I, except for primes that specialize to the whole ring.

Application 2.16 (Index of Stability)**.** *Let*

$$\operatorname{astab}(I)$$

denote the index of stability of the associated primes of the powers of I. Then

$$\operatorname{astab}(I) \leq \operatorname{astab}(\mathcal{E}_a).$$

Remark 2.17. The index of stability $\operatorname{astab}(I)$ is the least integer s such that

$$\operatorname{Ass}(R/I^r) = \operatorname{Ass}(R/I^s) \qquad \text{for all } r \geq s.$$

The inequality $\operatorname{astab}(I) \leq \operatorname{astab}(\mathcal{E}_a)$ expresses the extremal nature of $\mathcal{E}_a$: among square-free monomial ideals with a generators, it gives a universal upper bound for when associated primes stabilize.

Minimal Primes and Partitions

There is a correspondence

$$\{\text{partitions of } [a]\} \longleftrightarrow \{\text{minimal primes of } \mathcal{E}_a\}.$$

Example 2.18. For $a = 3$, the partition

$$\{\{1\}, \{2, 3\}\}$$

corresponds to the minimal prime

$$(Y_1, Y_{23}) \subseteq S_{[3]}.$$

Indeed, Y_1 divides ε_1, while Y_{23} divides both ε_2 and ε_3, so together Y_1 and Y_{23} meet every generator of $\mathcal{E}_3$.

Remark 2.19. A minimal prime of a square-free monomial ideal corresponds to a minimal vertex cover of its hypergraph of generators. For $\mathcal{E}_a$, the variables Y_A encode subsets $A \subseteq [a]$, and choosing variables whose indexing subsets partition $[a]$ gives a minimal way to cover all generators $\varepsilon_1, \dots, \varepsilon_a$.

Taylor Resolutions and Scarf Complexes

Example 2.20. Consider the square-free monomial ideal

$$I = (xy, yz, zw) \subseteq K[x, y, z, w].$$

Its Taylor complex has one vertex for each minimal generator and one face for each subset of the set of generators. The labels are given by least common multiples. For the generators

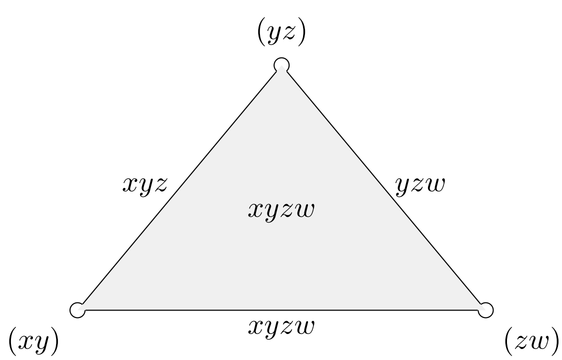
$$m_1 = xy, \qquad m_2 = yz, \qquad m_3 = zw,$$

the edge labels are

$$\operatorname{lcm}(xy, yz) = xyz, \qquad \operatorname{lcm}(yz, zw) = yzw, \qquad \operatorname{lcm}(xy, zw) = xyzw,$$

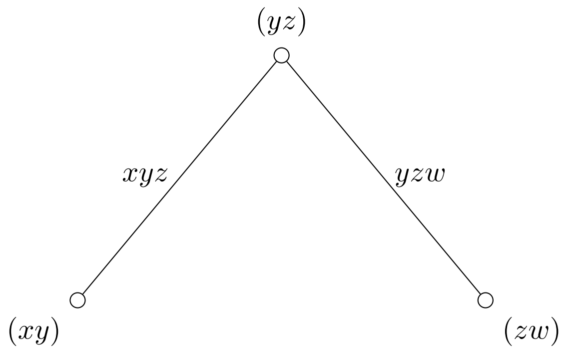
and the two-dimensional face is labeled

$$\operatorname{lcm}(xy, yz, zw) = xyzw.$$



Taylor complex

The Scarf complex keeps only those faces whose least common multiple label is unique. Since the edge $\{xy, zw\}$ and the two-dimensional face $\{xy, yz, zw\}$ both have label $xyzw$, those labels are not unique in the Taylor complex. The Scarf complex therefore consists of the two edges labeled xyz and yzw:



Scarf complex

Remark 2.21. The Taylor resolution supports a homological complex of free modules. For a monomial ideal minimally generated by three elements, the Taylor complex has the shape

$$0 \longrightarrow R \longrightarrow R^3 \longrightarrow R^3 \longrightarrow I \longrightarrow 0,$$

where the copy of R^3 closer to I corresponds to vertices, and the other copy of R^3 corresponds to edges. In the example above, the Taylor complex is generally not minimal because repeated least common multiple labels occur.

Remark 2.22. In the setting of the lecture, the ideals

$$\mathcal{E}_a, \qquad \mathcal{E}_a^2, \qquad \mathcal{E}_a^3$$

have Scarf complexes that support minimal free resolutions.

Remark 2.23. The Scarf complex is a subcomplex of the Taylor complex determined by uniqueness of multidegree labels. When it supports a minimal free resolution, it gives a much smaller resolution than the Taylor resolution, because it discards faces whose multidegrees occur more than once.

Comparison of Betti Number Bounds

The following table compares Betti numbers for Taylor resolutions, the best previously known bounds, and Scarf-complex bounds in the case $r = 3$ and $a = 6$.

$r = 3,\ a = 6$			
	Taylor	**Best known before**	**Scarf**
β_3	367,290	230,360	19,845
β_4	3,819,816	2,118,790	58,530
β_{20}	1.3×10^{15}	6.7×10^{13}	0

Remark 2.24. The dramatic reduction from the Taylor bounds to the Scarf bounds reflects the fact that the Taylor resolution is usually far from minimal. For the extremal ideals under discussion, Scarf complexes provide substantially sharper estimates for Betti numbers.

3 Lecture 3 — Tor, Complete Intersections & Minors by David Jorgensen

Rigidity of Tor

Theorem 3.1 (Auslander, 1960)**.** *Let M and N be finitely generated modules over an unramified regular local ring R. If*

$$\operatorname{Tor}_i^R(M, N) = 0,$$

then

$$\operatorname{Tor}_j^R(M, N) = 0 \qquad \text{for all } j \geq i.$$

Theorem 3.2 (Lichtenbaum, 1966)**.** *Rigidity of Tor holds for finitely generated modules over any regular local ring.*

Conjecture 3.3 (Peskine–Szpiro, 1972)**.** *Let M and N be finitely generated R-modules, and assume that*

$$\operatorname{pd}_R M < \infty.$$

Then

$$\operatorname{Tor}_i^R(M,N)=0 \quad\Longrightarrow\quad \operatorname{Tor}_j^R(M,N)=0 \qquad \textit{for all } j\geq i.$$

Equivalently, vanishing of one sufficiently relevant Tor module should force all higher Tor modules to vanish.

Theorem 3.4 (Heitmann, 1993)**.** *There exist an affine k-algebra R and finitely generated R-modules M,N such that*

$$\operatorname{pd}_R M=2, \qquad \operatorname{length}_R(N)=3,$$

with

$$\operatorname{Tor}_1^R(M,N)=0 \qquad \textit{but} \qquad \operatorname{Tor}_2^R(M,N)\neq 0.$$

Heitmann's generic-resolution example

Specifically, consider a generic resolution

$$0\longrightarrow R^2 \xrightarrow{X} R^4 \xrightarrow{Y} R^8.$$

In this example,

$$\operatorname{length}_R(M\otimes_R N)=1.$$

Furthermore,

$$\operatorname{im}(X\otimes_R N)=\mathfrak{m}N^4,$$
$$\operatorname{im}(Y\otimes_R N)=\mathfrak{m}N^8.$$

Tensoring the displayed complex with N gives

$$0\longrightarrow N^2 \xrightarrow{X\otimes_R N} N^4 \xrightarrow{Y\otimes_R N} N^8.$$

Since $\operatorname{length}_R(N)=3$, the lengths of the three nonzero free N-modules are

$$\operatorname{length}_R(N^2)=6, \qquad \operatorname{length}_R(N^4)=12, \qquad \operatorname{length}_R(N^8)=24.$$

The length-counting structure may be represented as follows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & N^2 & \xrightarrow{X\otimes_R N} & N^4 & \xrightarrow{Y\otimes_R N} & N^8 \\
 & & \text{length}=6 & & \text{length}=12 & & \text{length}=24 \\
 & & \text{not exact} & & \text{exact by counting lengths} & &
\end{array}$$

Remark 3.5. The displayed complex records the failure of full Tor-rigidity in Heitmann's construction. The phrase "exact by counting lengths" refers to the fact that, after tensoring with N, one can compare the lengths of kernels and images to verify exactness at the indicated term. The failure of exactness at the preceding term is what produces nonvanishing of the relevant Tor module.

Tor over Complete Intersections

Theorem 3.6 (Jorgensen, 1995)**.** *Let R be a complete intersection, and let M, N be finitely generated R-modules. Set*

$$r = \operatorname{cx}_R(M, N).$$

If

$$\operatorname{Tor}_i^R(M, N) = 0$$

for $r + 1$ consecutive values of $i > \dim R$, then

$$\operatorname{Tor}_i^R(M, N) = 0 \qquad \textit{for all } i > \dim R.$$

Remark 3.7. Here $\operatorname{cx}_R(M, N)$ denotes the complexity of the pair (M, N), measuring the polynomial growth rate of the sequence of lengths or ranks associated to $\operatorname{Tor}_i^R(M, N)$, when such a formulation is available. Over complete intersections, complexity controls the amount of consecutive vanishing needed to force eventual vanishing.

A next logical step

Theorem 3.8 (Bergh–Jorgensen–Thompson, 2024)**.** *Let $(\mathcal{T}, \Sigma)$ be a triangulated category admitting a central ring action from a graded ring*

$$R = R^0[R^d],$$

and let X, Y be objects of $\mathcal{T}$. Assume that

$$\operatorname{Hom}_{\mathcal{T}}^{\geq 0}(X, Y)$$

is finitely generated over R. Assume, in addition, that one of the following conditions holds:

(1) $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ *has finite length over R^0 for all $n \gg 0$.*

(2) R^0 *contains an infinite field, or R^0 is local with infinite residue field.*

Then there exists an integer $m_0 \in \mathbb{Z}$ such that if

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{n_i} Y) = 0$$

for a complete set of residues

$$n_1, \ldots, n_d \geq m_0 \pmod{d},$$

then

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{n} Y) = 0 \qquad \text{for all } n \geq m_0.$$

Remark 3.9 (Bergh–Jorgensen–Thompson). One may introduce a multiplicity

$$e(X, Y)$$

in the triangulated category $\mathcal{T}$. If $e(X, Y) = 0$, then the required vanishing hypothesis can be weakened to a "half-vanishing" condition on the groups

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{n} Y).$$

Tor Persistence

Tor persistence asks whether eventual vanishing of self-Tor forces finite projective dimension:

$$\mathrm{Tor}_i^R(M, M) = 0 \text{ for all } i \gg 0 \quad \Longrightarrow \quad \mathrm{pd}_R\, M < \infty.$$

Theorem 3.10 (Asani, 2024)**.** *Let X be an $r \times c$ matrix of indeterminates with $r \leq c$, and set*

$$R = K[X]/I_r(X).$$

Let

$$M = \operatorname{coker} X.$$

Then

$$\mathrm{Tor}_2^R(M, M) \neq 0.$$

Remark 3.11. Here $I_r(X)$ denotes the ideal generated by the maximal minors of the generic $r \times c$ matrix X. The theorem gives a nonvanishing result for the self-Tor of the cokernel module over a determinantal hypersurface-like quotient.

Complete Intersections

Definition 3.12. A local ring R is a complete intersection if

$$R = Q/I,$$

where Q is a regular local ring and I is generated by a regular sequence.

Definition 3.13 (Grothendieck)**.** A local ring R is a complete intersection if

$$\widehat{R} = Q/I,$$

where Q is a regular local ring and I is generated by a regular sequence.

Question 3.14 (Avramov). Are complete intersections complete intersections?

The answer is: sometimes.

Theorem 3.15 (Heitmann–Jorgensen)**.** *Let R be a one-dimensional integral domain. If R is a complete intersection after completion, then R is a complete intersection.*

Theorem 3.16 (Heitmann–Jorgensen)**.** *There exists a three-dimensional local ring R such that*

$$\widehat{R} = Q/I,$$

where Q is a regular local ring and I is generated by a regular sequence, but R is not a homomorphic image of a regular local ring. In particular, R is not excellent.

Remark 3.17. The distinction here is between being a complete intersection intrinsically and becoming one after completion. Grothendieck's formulation uses the completed local ring $\widehat{R}$, while the more rigid formulation asks for R itself to be presented as a quotient of a regular local ring by a regular sequence.

Minors

Buchsbaum–Eisenbud

Theorem 3.18 (Buchsbaum–Eisenbud, 1974)**.** *Let R be Noetherian, and let*

$$0 \longrightarrow F_n \xrightarrow{A_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{A_1} F_0$$

be a finite free resolution. For a matrix A, the ideal $I_r(A)$ is generated by the $r \times r$ minors of A. Equivalently, one chooses r rows and r columns and takes the corresponding determinant:

$$I_r(A) = \Big(\det A_{\alpha,\beta} \;\Big|\; \alpha \subseteq \textit{rows of } A,\ \beta \subseteq \textit{columns of } A,\ |\alpha| = |\beta| = r\Big).$$

The source notes record this schematically as

$$I_r(A) = \begin{bmatrix} c \\ o \\ l \\ u \\ m \\ n \end{bmatrix} \cdot \begin{bmatrix} r & o & w \end{bmatrix}.$$

Remark 3.19. The schematic column-vector times row-vector notation is a mnemonic for selecting columns and rows in the construction of minors. In the usual notation, $I_r(A)$ denotes the ideal generated by all $r \times r$ minors of the matrix A.

A determinantal identity

Let X be an $r \times c$ matrix. For a positive integer d, index the $d \times d$ minors by pairs

$$(\alpha, \beta),$$

where α is a d-element subset of the row indices and β is a d-element subset of the column indices. Write

$$M_{\alpha\beta}$$

for the corresponding $d \times d$ minor.

Theorem 3.20 (Heitmann, 2024)**.** *Let R be a ring, not necessarily Noetherian, and let X be an $r \times c$ matrix of rank d. Assume that $M_{\mu\nu}$ is a regular $d \times d$ minor. Then*

$$M_{\mu\nu} M_{\alpha\beta} = M_{\alpha\nu} M_{\mu\beta} \qquad \textit{for all } \alpha, \beta.$$

Equivalently,

$$M_{\mu\nu}\begin{bmatrix} M_{\alpha_1\beta_1} & \cdots & M_{\alpha_1\beta_{\binom{c}{d}}} \\ \vdots & & \vdots \\ M_{\alpha_{\binom{r}{d}}\beta_1} & \cdots & M_{\alpha_{\binom{r}{d}}\beta_{\binom{c}{d}}} \end{bmatrix} = \begin{bmatrix} M_{\alpha_1\nu} \\ \vdots \\ M_{\alpha_{\binom{r}{d}}\nu} \end{bmatrix} \begin{bmatrix} M_{\mu\beta_1} & \cdots & M_{\mu\beta_{\binom{c}{d}}} \end{bmatrix}.$$

Remark 3.21. The displayed matrix identity says that, after multiplying the full matrix of $d \times d$ minors by the fixed regular minor $M_{\mu\nu}$, the result has rank one: it factors as a column vector times a row vector. Entrywise, this is precisely the identity

$$M_{\mu\nu}M_{\alpha\beta} = M_{\alpha\nu}M_{\mu\beta}.$$

The regularity assumption on $M_{\mu\nu}$ is essential for cancellation-type arguments in rings with zero divisors.

4 Lecture 4 — Heitmann type constructions by Susan Loepp

Finite Posets Inside Spectra

Definition 4.1. Let X be a finite poset and let R be a ring. An embedding

$$\iota\colon X \hookrightarrow \operatorname{Spec}(R)$$

is said to *preserve saturated chains* if, whenever

$$x_0 < x_1 < \cdots < x_r$$

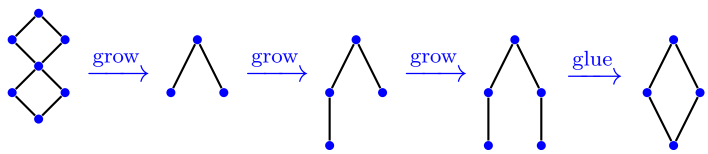
is a saturated chain in X, the corresponding chain

$$\iota(x_0) \subsetneq \iota(x_1) \subsetneq \cdots \subsetneq \iota(x_r)$$

is saturated inside the image of X in $\operatorname{Spec}(R)$. Equivalently, no additional element of the embedded copy of X lies strictly between two consecutive primes in such a chain.

Theorem 4.2 (Finite posets in spectra of quasi-excellent domains)**.** *Let X be a finite poset. Then there exists a quasi-excellent domain R such that X can be embedded into* $\operatorname{Spec}(R)$ *in a way that preserves saturated chains.*

Remark 4.3. The theorem says that finite order-theoretic data can be realized inside the prime spectrum of a very well-behaved Noetherian domain. The phrase "preserves saturated chains" is essential: it asserts that the covering relations of the poset are reflected by saturated inclusions among the corresponding prime ideals.



Growing Minimal Nodes

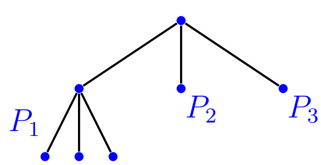
Proposition 4.4 (Growing)**.** *Suppose X is a finite poset and R is a quasi-excellent local ring such that X can be embedded into* $\operatorname{Spec}(R)$ *in a way that preserves saturated chains. Let Y be a poset obtained from X by growing finitely many nodes out of a minimal node. Then there exists a quasi-excellent local ring S such that Y can be embedded into* $\operatorname{Spec}(S)$ *in a way that preserves saturated chains.*

Remark 4.5. Let R be quasi-excellent. Consider the formal power series ring

$$R[[Y, Z]].$$

The original poset X "lives" in $R[[Y, Z]]$ above the ideal $\langle Y, Z\rangle$.

Remark 4.6. In the phrase $R[[Y, Z]]$, the symbols Y and Z denote formal variables. This is separate from the notation Y used above for the enlarged poset. The geometric idea is that passing from R to a formal power series ring gives enough room to separate several new branches over a chosen minimal prime by imposing distinct linear equations such as $Y - \beta_i Z$.



Construction 4.7 (A typical growing ideal)**.** *Let P_1, P_2, P_3 be the relevant primes from the original embedded poset, and choose distinct scalars $\beta_1, \beta_2, \beta_3$. In the power series*

ring $R[[Y,Z]]$, one considers the ideal

$$J = \langle P_2, Y, Z\rangle \cap \langle P_3, Y, Z\rangle \cap \langle P_1, Y - \beta_1 Z\rangle \cap \langle P_1, Y - \beta_2 Z\rangle \cap \langle P_1, Y - \beta_3 Z\rangle.$$

Remark 4.8. The components $\langle P_2, Y, Z\rangle$ and $\langle P_3, Y, Z\rangle$ keep the branches over P_2 and P_3 fixed above the closed formal fiber defined by $\langle Y, Z\rangle$. The components $\langle P_1, Y - \beta_i Z\rangle$ split the branch over P_1 into several distinguishable directions. The use of distinct β_i's ensures that these directions remain separated.

Gluing Minimal Nodes

Proposition 4.9 (Gluing)**.** *Suppose X is a finite poset and R is a quasi-excellent ring such that X can be embedded into* $\mathrm{Spec}(R)$ *in a way that preserves saturated chains. Let Y be a poset obtained from X by gluing some minimal nodes together. Then there exists a quasi-excellent ring S such that Y can be embedded into* $\mathrm{Spec}(S)$ *in a way that preserves saturated chains.*

Remark 4.10 (Notes on gluing). Let

$$Z = \{P_1, \dots, P_n\}$$

be the set of minimal primes of R that one wants to glue together. Then the construction produces a subring S with the following properties:

(1)
$$P_i \cap S = P_1 \cap S \qquad \text{for all } i = 1, \dots, n.$$

(2) The map

$$\mathrm{Spec}(R) \setminus Z \longrightarrow \mathrm{Spec}(S) \setminus \{P_1 \cap S\}, \qquad P \longmapsto P \cap S$$

is a bijection.

Remark 4.11. The first condition says that the chosen minimal primes become indistinguishable after contraction to S. The second condition says that no other part of the spectrum is changed: away from the glued primes, contraction gives a one-to-one correspondence of prime ideals.

Embedding Finite Posets into Spectra of UFDs

Theorem 4.12 (Finite posets inside spectra of Noetherian UFDs)**.** *Let X be a finite poset. Then there exists a Noetherian UFD A such that X can be embedded into* $\mathrm{Spec}(A)$ *in a way that preserves saturated chains.*

Lecture notes sketch. We know that X can be embedded into $\mathrm{Spec}(R)$ for some Noetherian domain R. Let

$$T = R[[x_1, \dots, x_n]]$$

where $n \geq \dim X$. Then X "lives" above the ideal

$$\langle x_1, \dots, x_n \rangle$$

because

$$T/\langle x_1, \dots, x_n \rangle \cong R.$$

Here R is viewed as a subring of T through the natural coefficient embedding. Now use a Heitmann-type construction to construct A, a local UFD such that

$$\widehat{A} = T.$$

The construction is arranged so that the relevant prime ideals of T, which encode the embedded copy of X, are contractions of prime ideals from A. Thus the embedded copy of X descends to $\mathrm{Spec}(A)$, still preserving saturated chains. Finally, the construction gives

$$\dim A \geq 2.$$

□

Remark 4.13. The key structural input is that certain complete local rings arise as completions of local UFDs. This is the Heitmann-type mechanism: start with a complete local ring T, then construct a local UFD A whose completion is T. Since $\mathrm{Spec}(T)$ controls formal neighborhoods of $\mathrm{Spec}(A)$, carefully chosen prime ideals in T can be made to reflect prime ideals in A.

A Dimension-Sharp Local UFD Characterization

Theorem 4.14 (Dimension-sharp realization by local UFDs)**.** *Let X be a finite poset. Then there exists a local UFD A such that*

$$\dim A = \dim X$$

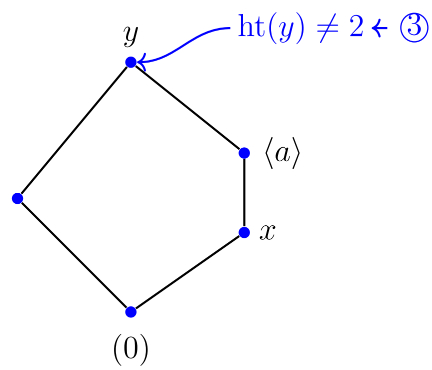
and X can be embedded into $\mathrm{Spec}(A)$ *in a way that preserves saturated chains if and only if the following three conditions hold:*

(1) *X has exactly one maximum element.*

(2) *X has exactly one minimum element.*

(3) *If $x \in X$ has height* 1 *and $x < y$ is saturated, then* $\mathrm{ht}(y) = 2$.

Remark 4.15. The uniqueness of the maximum element reflects the fact that a local ring has a unique maximal ideal. The uniqueness of the minimum element reflects the fact that a domain has a unique minimal prime, namely (0). Condition (3) reflects the principal ideal theorem in a UFD: height-one primes are principal, and a saturated immediate specialization above a height-one prime cannot jump directly to height greater than 2.



Example 4.16 (Failure of condition (3))**.** The diagram above illustrates a finite poset with a unique minimum element (0) and a unique maximum element y, but with a saturated relation

$$x < y$$

where x has height 1 and $\mathrm{ht}(y) \neq 2$. In a local UFD, a height-one prime has the form $\langle a \rangle$ for some prime element a, and an immediate saturated prime above it must have height 2. Thus the displayed configuration violates condition (3).

5 Lecture 5 — Closure Operations in Characteristic 0 induced by resolutions of rings by Karl Schwede

History of Singularities

The guiding philosophy is that singularities in characteristic zero often have characteristic $p > 0$ analogues after reduction modulo p. The following table records the standard dictionary discussed in the notes.

Characteristic 0	**Characteristic $p > 0$**
Log terminal	Weakly F-regular, or strongly F-regular
Rational singularities	F-rational singularities

Here weak F-regularity means that all ideals are tightly closed, while F-rationality means that all parameter ideals are tightly closed.

Theorem 5.1 (Smith, Hara, Mehta–Srinivas, Watanabe)**.** *Let $R_{\mathbb{Z}}$ be a ring of finite type over $\mathbb{Z}$. Define*

$$R_{\mathbb{Q}} = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad R_p = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Then the following hold.

(1) $R_{\mathbb{Q}}$ has rational singularities if and only if R_p is F-rational for all sufficiently large primes p.

(2) If $R_{\mathbb{Q}}$ is $\mathbb{Q}$-Gorenstein, then $R_{\mathbb{Q}}$ is log terminal if and only if R_p is strongly F-regular for all sufficiently large primes p.

Tight Closure in Characteristic $p > 0$

Definition 5.2 (Hochster–Huneke tight closure)**.** Let R be a domain of characteristic $p > 0$, and let $J \subseteq R$ be an ideal. An element $x \in R$ belongs to the tight closure J^* of J if there exists an element $0 \neq c \in R$ such that, for all $e \gg 0$,

$$c^{1/p^e} x \in J R^{1/p^e}$$

under the Frobenius-root extension $R \to R^{1/p^e}$. Equivalently, after applying the p^e-th power map, this condition may be written as

$$c x^{p^e} \in J^{[p^e]} \qquad \text{for all } e \gg 0,$$

where $J^{[p^e]}$ denotes the ideal generated by the p^e-th powers of elements of J.

Remark 5.3. If $J = (f_1, \dots, f_s)$, then

$$J^{[p^e]} = (f_1^{p^e}, \dots, f_s^{p^e}).$$

The use of R^{1/p^e} is equivalent to the usual Frobenius formulation when R is reduced, and it emphasizes that tight closure tests membership after adjoining p^e-th roots.

Basic Properties of Tight Closure

The following are the central formal properties of tight closure relevant to the construction of closure operations.

(0) If R is regular, then $J^* = J$ for every ideal $J \subseteq R$.

(1) **Colon capturing.** Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \dots, x_d$ be a system of parameters. Then, for each $i < d$,

$$(x_1, \dots, x_i) : x_{i+1} \subseteq (x_1, \dots, x_i)^*.$$

The multiplication map

$$R/(x_1, \dots, x_i) \xrightarrow{\cdot x_{i+1}} R/(x_1, \dots, x_i)$$

has kernel

$$\frac{(x_1, \dots, x_i) : x_{i+1}}{(x_1, \dots, x_i)}.$$

Thus colon capturing says that this kernel is controlled by tight closure.

(2) If R is complete local, then

$$x \in J^* \iff \text{there exists a big Cohen–Macaulay } R\text{-algebra } B \text{ such that } x \in JB.$$

(3) **Behavior under finite extensions.** If $R \subseteq S$ is a finite extension, then

$$(JS)^* \cap R = J^*.$$

(4) **Briançon–Skoda theorem.** If $J = (f_1, \dots, f_n)$ is generated by n elements, then

$$\overline{J^{n+k-1}} \subseteq (J^k)^*$$

for all $k \geq 1$.

A Mixed Characteristic Closure of Heitmann

Definition 5.4 (Heitmann-type closure)**.** Let R be a mixed characteristic domain and let $J \subseteq R$ be an ideal. One defines a closure operation by declaring that an element $x \in R$ belongs to the closure of J if

$$c^{\varepsilon} x \in \bigcap_n (J, p^n) R^+ \qquad \text{for all } \varepsilon > 0.$$

Remark 5.5. Here R^+ denotes the absolute integral closure of R. The expression c^{ε} is the usual "almost mathematics" notation appearing in Heitmann's construction.

Characteristic Zero Operations from Resolutions

Let $(R, \mathfrak{m})$ be a domain that is excellent and admits a dualizing complex. Let

$$\pi : Y \longrightarrow \operatorname{Spec}(R)$$

be a resolution of singularities, which exists in characteristic zero by Hironaka's theorem. Thus Y is nonsingular.

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \\ \operatorname{Spec}(R) \end{tikzcd}$$

The derived pushforward $R\pi_* \mathcal{O}_Y$, or equivalently the derived global sections $R\Gamma(\mathcal{O}_Y)$ in the local affine setting, acts as a Cohen–Macaulay complex, or dg-algebra, associated to the resolution. In particular, if $d = \dim R$, then the local cohomology of this complex satisfies

$$H^i_{\mathfrak{m}}(R\Gamma(\mathcal{O}_Y)) = 0 \qquad \text{for all } i < d,$$

while

$$H^d_{\mathfrak{m}}(R\Gamma(\mathcal{O}_Y)) \neq 0.$$

Remark 5.6. The vanishing above is the derived analogue of the Cohen–Macaulay property. Instead of requiring R itself to be Cohen–Macaulay, one replaces R by a complex coming from a resolution of singularities. This is the mechanism by which characteristic-zero geometry produces closure operations resembling tight closure.

The Hironaka Closure

Let $J \subseteq R$ be an ideal. Define

$$J^{\mathrm{Hir}} = \ker\left(R \longrightarrow H_0\left((R/J) \otimes_R^L R\Gamma(\mathcal{O}_Y)\right)\right).$$

Equivalently, $x \in J^{\mathrm{Hir}}$ if the image of x in

$$H_0\left((R/J) \otimes_R^L R\Gamma(\mathcal{O}_Y)\right)$$

vanishes.

Question 5.7. Is the Hironaka closure idempotent? That is, does one always have

$$(J^{\mathrm{Hir}})^{\mathrm{Hir}} = J^{\mathrm{Hir}}?$$

The Koszul–Hironaka Closure

Let $J = (f_1, \dots, f_c)$. Using the Koszul complex $\mathrm{Kos}(f; R)$, define

$$J^{\mathrm{KH}} = \ker\left(R \longrightarrow H_0\left(\mathrm{Kos}(f; R) \otimes_R^L R\Gamma(\mathcal{O}_Y)\right)\right).$$

This construction gives a closure operation satisfying

$$(J^{\mathrm{KH}})^{\mathrm{KH}} = J^{\mathrm{KH}}.$$

Remark 5.8. The Koszul complex remembers the chosen generators of J homologically. Replacing R/J by $\mathrm{Kos}(f; R)$ improves the formal behavior of the construction and yields idempotence for the resulting closure operation.

Properties of the Koszul–Hironaka Closure

The Koszul–Hironaka closure has the following properties.

(0) R has rational singularities if and only if all ideals are Hir-closed. Moreover, one direction is implied by the condition that one full parameter ideal is KH-closed.

(1) **Colon capturing.** For a system of parameters $x_1, \dots, x_d$,

$$(x_1, \dots, x_i)^{\mathrm{KH}} : x_{i+1} \subseteq (x_1, \dots, x_i)^{\mathrm{KH}}.$$

More generally,

$$\left((x_1^a, x_2, \dots, x_i)^{\mathrm{KH}} : x_i^t\right) \subseteq (x_1^{a-t}, x_2, \dots, x_i)^{\mathrm{KH}}.$$

(2) **Finite extensions.** If $R \subseteq S$ is a finite extension, then

$$(JS)^{\mathrm{KH}} \cap R = J^{\mathrm{KH}}.$$

(3) **Briançon–Skoda type containment.** If $J = (f_1, \dots, f_n)$, then

$$\overline{J^n} \subseteq J^{\mathrm{KH}}.$$

(4) The closure is computable in Macaulay2.

(5) The closure operation localizes.

General Closure Operations

A closure operation on ideals is a rule

$$I \longmapsto I^*$$

satisfying the following formal properties:

- $I \subseteq I^*$;
- $I^{**} = I^*$;
- if $I \subseteq J$, then $I^* \subseteq J^*$.

A natural question is whether closure commutes with localization. If $W \subseteq R$ is a multiplicatively closed set and $S = W^{-1}R$, one asks whether

$$I^S = (IS)^*.$$

This is false for tight closure in general.

Closure Operations from Modules

Let M be an R-module. One can define a closure-like operation on ideals by setting

$$I^c = \operatorname{ann}_R(M/IM).$$

Equivalently,

$$\begin{aligned} I^c &= \{x \in R \mid xM \subseteq IM\} \\ &= \operatorname{ann}_R(R/I \otimes_R M) \\ &= \operatorname{ann}_R\Big(\operatorname{coker}(I \hookrightarrow R) \otimes_R M\Big). \end{aligned}$$

If $N \subseteq X$, then there is an induced map

$$N \otimes_R M \longrightarrow X \otimes_R M.$$

Colon Capturing from Maximal Cohen–Macaulay Modules

Assume M is a maximal Cohen–Macaulay R-module. The ring R itself need not be Cohen–Macaulay. Define

$$I^c = \operatorname{ann}_R(M/IM).$$

Let $x_1, \ldots, x_d$ be a system of parameters on R. Since M is maximal Cohen–Macaulay, the sequence $x_1, \ldots, x_d$ is M-regular. Then

$$(x_1, \ldots, x_i) : x_{i+1} \subseteq (x_1, \ldots, x_i)^c.$$

> *Remark* 5.9. Equality holds when $x_1, \ldots, x_d$ is a regular sequence on R.

Proof. Assume

$$ux_{i+1} \in (x_1, \ldots, x_i).$$

Then

$$ux_{i+1}M \subseteq (x_1, \ldots, x_i)M.$$

Since M is Cohen–Macaulay, the sequence $x_1, \ldots, x_d$ is regular on M. Hence multiplication by x_{i+1} is injective on

$$M/(x_1, \ldots, x_i)M.$$

Therefore

$$uM \subseteq (x_1, \ldots, x_i)M.$$

By the definition of c-closure,

$$u \in (x_1, \ldots, x_i)^c.$$

This proves the desired containment. □

Thus, if a closure operation has colon capturing and satisfies

$$(x_1, \ldots, x_i)^c = (x_1, \ldots, x_i)$$

for parameter ideals, then R is Cohen–Macaulay.

> *Remark* 5.10. This is the conceptual reason tight closure detects Cohen–Macaulayness. Tight closure has colon capturing, and in regular rings all ideals are tightly closed. Hence tight closure behaves as though it were produced from a Cohen–Macaulay object, namely R^+, the absolute integral closure.

The Hochster–Roberts Theorem

Theorem 5.11 (Hochster–Roberts, Noetherian direct summand theorem)**.** *If $R \to S$ is a pure extension of Noetherian rings and S is regular, then R is Cohen–Macaulay. In particular, a Noetherian direct summand of a regular ring is Cohen–Macaulay.*

Example 5.12. Let

$$R = k[x^3, x^2y, xy^2, y^3] \subseteq S = k[x, y].$$

This is a typical invariant-theoretic or Veronese-type example of a subring of a regular ring.

Proof idea via tight closure. It is enough to prove that $I^* = I$ for every ideal $I \subseteq R$. This implies, through colon capturing, that every system of parameters is a regular sequence; hence R is Cohen–Macaulay.

Let $x \in I^*$. Then

$$x \in (IS)^* \cap R.$$

Indeed, if $cx^q \in I^{[q]}$ for all large $q = p^e$, then also

$$cx^q \in (IS)^{[q]}$$

for all large q. Since S is regular, all ideals of S are tightly closed, so

$$(IS)^* = IS.$$

Therefore

$$x \in IS \cap R.$$

Because $R \to S$ is pure, and in particular split in the direct summand case, write

$$S \cong R \oplus N$$

as R-modules. Then

$$S/IS \cong R/I \oplus N/IN.$$

Consequently, the kernel of the natural map

$$R \longrightarrow S/IS$$

is exactly I. Equivalently,

$$IS \cap R = I.$$

Thus $x \in I$. Hence $I^* = I$ for all ideals I, and R is Cohen–Macaulay. $\square$

Reduction to Characteristic p

Let X be a variety over a field of characteristic zero. Suppose

$$I \subseteq \mathbb{C}[x_1, \ldots, x_n] = S$$

and

$$X = \operatorname{Spec}(S/I).$$

Since I has finitely many generators, only finitely many complex coefficients occur. Denote these coefficients by

$$\alpha_1, \ldots, \alpha_N.$$

Choose a finitely generated $\mathbb{Z}$-subalgebra

$$A_0 \subseteq \mathbb{C}$$

containing all the coefficients $\alpha_1, \ldots, \alpha_N$, and set

$$R_{A_0} = \frac{A_0[x_1, \ldots, x_n]}{I_{A_0}},$$

where I_{A_0} is generated by the same polynomials, now viewed as polynomials with coefficients in A_0. Then

$$R_{A_0} \otimes_{A_0} \mathbb{C} \cong S/I.$$

For a prime $\mathfrak{p} \in \operatorname{Spec}(A_0)$, the fiber

$$R_{\mathfrak{p}} := R_{A_0} \otimes_{A_0} \kappa(\mathfrak{p})$$

is a reduction of X to characteristic p, where p is the residue characteristic of $\kappa(\mathfrak{p})$.

One then studies whether a given property holds for $R_{\mathfrak{p}}$ for all sufficiently general, or often sufficiently large, primes p. Equivalently, one may construct a finitely generated $\mathbb{Z}$-algebra

$$A = \frac{\mathbb{Z}[\alpha_1, \ldots, \alpha_N][x_1, \ldots, x_n]}{I}.$$

If P_X is a statement about X, the philosophy of reduction modulo p compares

$$P(A \otimes_{\mathbb{Z}} \mathbb{Q}) \qquad \text{and} \qquad P(A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}).$$

For many geometric and singularity-theoretic properties, this comparison holds for a dense set of closed points of $\operatorname{Spec}(\mathbb{Z}[\alpha_1, \ldots, \alpha_N])$, or for all sufficiently large primes after making suitable choices of model.

Remark 5.13. In practice, one must often localize the finitely generated $\mathbb{Z}$-algebra A_0, replacing it by $A_0[g^{-1}]$, to ensure good behavior of the family. This process removes finitely many bad primes and is the technical meaning of the phrase "for all sufficiently general primes."

Frobenius and Regular Rings

In characteristic $p > 0$, the Frobenius map is

$$F : R \longrightarrow R, \qquad x \longmapsto x^p.$$

The binomial theorem simplifies because

$$(x + y)^p = x^p + y^p.$$

Thus F is a ring homomorphism. For example, if

$$R = k[x, y],$$

then the image of Frobenius is the subring

$$k[x^p, y^p] \subseteq k[x, y].$$

Theorem 5.14 (Kunz). *A Noetherian ring R of characteristic $p > 0$ is regular if and only if the Frobenius map $F : R \to R$ is flat.*

Remark 5.15. Kunz's theorem explains why regular rings have trivial tight closure. Flatness of Frobenius prevents new containment relations from appearing after applying Frobenius powers, so the condition $cx^{p^e} \in I^{[p^e]}$ forces $x \in I$ in regular rings.