PASCA 2.0

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Lecture Notes

Singularities in Different Characteristics

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Contents

1 Lecture-1 May 18, 2026

Let $(R, \mathfrak{m})$ be a Noetherian local domain. The goal is to study the singularities of R. One general way to probe singularities is to compare R with a better ring R' via an inclusion

$$R \hookrightarrow R',$$

or geometrically to study a morphism

$$X \longrightarrow \operatorname{Spec} R.$$

Another variant is the following. Let A be a regular local ring and suppose there is a surjection

$$A \twoheadrightarrow R = A/I.$$

Then one wants to understand the ideal I in terms of maps

$$A \longrightarrow A'.$$

Throughout this lecture, assume that

$$\operatorname{char}(R) = p > 0.$$

The Frobenius endomorphism is the ring map

$$\begin{aligned} F : R &\longrightarrow R, \\ r &\longmapsto r^p. \end{aligned}$$

Remark 1.1. The Frobenius map is special to characteristic p. It is additive because

$$(a+b)^p = a^p + b^p$$

in characteristic p, and it is multiplicative for formal reasons. Thus it is a ring endomorphism.

One issue is that the source and target of F are the same ring. A useful way to separate them is to adjoin p-th roots. Define

$$R^{1/p} = \{x \in \overline{K(R)} \mid x^p \in R\},$$

where $K(R)$ is the fraction field of R and $\overline{K(R)}$ is an algebraic closure. Thus $R^{1/p}$ is the ring of p-th roots of elements of R. The Frobenius map on R is identified with the p-th power map on $R^{1/p}$ through the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r, "F"] \arrow[d, equal] & R \arrow[d, "\simeq"'] \\
R \arrow[r, hook] & R^{1/p}.
\end{tikzcd}$$

Remark 1.2. Although R and $R^{1/p}$ are isomorphic as abstract rings when R is reduced, they are not the same as R-modules. The R-module structure on $R^{1/p}$ is given by

$$r \cdot s^{1/p} = (r^p s)^{1/p}.$$

This distinction is the point of Kunz's theorem.

Kunz's Theorem

Theorem 1.3 (Kunz)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$. Then R is regular if and only if the Frobenius map*

$$F : R \to R$$

is flat. Equivalently, in the reduced/domain notation used above, R is regular if and only if $R^{1/p}$ is flat as an R-module.

Remark 1.4. As rings, R and $R^{1/p}$ are isomorphic in the reduced setting. The content of Kunz's theorem is about the R-module structure induced by Frobenius.

Example 1.5 (Polynomial and regular local rings)**.** Let

$$R = k[x_1, \dots, x_n]_{\mathfrak{m}}.$$

Then

$$R^{1/p} = k^{1/p}[x_1^{1/p}, \dots, x_n^{1/p}]_{\mathfrak{m}^{1/p}}.$$

When k is perfect, $R^{1/p}$ is a free R-module with basis

$$\left\{ x_1^{a_1/p} \cdots x_n^{a_n/p} \;\middle|\; 0 \le a_i \le p-1 \right\}.$$

The first few basis elements look like

$$1, \quad x_1^{1/p}, \quad \dots, \quad x_n^{1/p}, \quad x_1^{1/p}x_2^{1/p}, \quad \dots, \quad (x_1 \cdots x_n)^{(p-1)/p}.$$

More generally, if $[k^{1/p} : k] < \infty$, then one also includes a k-basis of $k^{1/p}$. If $[k^{1/p} : k] = \infty$ and R was the power series ring instead, $R^{1/p}$ is flat but not a free R-module. The polynomial ring does continue to be free but not finite.

Sketches of Proofs of Kunz's Theorem

First sketch: reduction to the complete case

One proof of Kunz's theorem proceeds by reducing to the complete local case. After completion, one uses the Cohen Structure Theorem . In the regular complete case, one has

$$R \cong k[[x_1, \dots, x_n]]$$

or a closely related coefficient-field variant. In this presentation, the behavior of Frobenius can be checked explicitly using p-th root monomials.

Remark 1.6. Completion is faithfully flat, so many properties relevant to Kunz's theorem, such as regularity and flatness of Frobenius, may be checked after completing. The Cohen Structure Theorem then gives explicit coordinates.

Second sketch: Auslander–Buchsbaum

Assume R is regular and $R^{1/p}$ is finite as an R-module; in other words, assume R is F-finite. By the Auslander–Buchsbaum formula ,

$$\operatorname{pdim}_R(R^{1/p}) + \operatorname{depth}_R(R^{1/p}) = \operatorname{depth}(R) = \dim(R). \tag{$*$}$$

Here we use that a regular local ring is Cohen–Macaulay, so

$$\operatorname{depth}(R) = \dim(R).$$

Suppose

$$\mathfrak{m} = (x_1, \dots, x_d), \qquad d = \dim R.$$

Since R is regular, $x_1, \dots, x_d$ is a regular system of parameters; in particular, it is an R-regular sequence. It follows that

$$x_1^p, \dots, x_d^p$$

is also an R-regular sequence. Under the identification

$$R \xrightarrow{\cong} R^{1/p},$$

the elements x_i acting on $R^{1/p}$ correspond to the elements x_i^p acting on R. Hence $x_1, \ldots, x_d$ form an $R^{1/p}$-regular sequence. Therefore

$$\operatorname{depth}_R(R^{1/p}) = \dim R.$$

Substituting this into $(*)$, we obtain

$$\operatorname{pdim}_R(R^{1/p}) = 0.$$

Thus $R^{1/p}$ is projective as an R-module, and hence flat.

Remark 1.7. The condition $\operatorname{pdim}_R(R^{1/p}) = 0$ means precisely that $R^{1/p}$ is projective. Since R is local, every finitely generated projective R-module is free. In particular, it is flat.

Third sketch: the local criterion for flatness

Another proof views

$$R \longrightarrow R^{1/p}$$

as a local homomorphism of Noetherian rings. One checks the following conditions:

(1) R is regular.

(2) The dimension formula holds:

$$\dim(R^{1/p}) = \dim R + \dim\Big(R^{1/p}/\mathfrak{m}R^{1/p}\Big).$$

(3) $R^{1/p}$ is Cohen–Macaulay.

These conditions imply that $R^{1/p}$ is flat over R.

Remark 1.8. This is an application of a flatness criterion for local maps. Roughly, a Cohen–Macaulay target over a regular source is flat when the dimensions of the source, target, and closed fiber match correctly.

F-Split and Strongly F-Regular Rings

Assume that $R^{1/p}$ is a free R-module with some basis. If there exists a projection onto a direct summand

$$\rho : R^{1/p} \longrightarrow R$$

such that

$$\rho(c^{1/p}) = 1$$

for some $c \in R$, then the composite

$$R \xrightarrow{\cdot c^{1/p}} R^{1/p} \xrightarrow{\rho} R$$

sends 1 to 1. This is represented by the diagram

$$\begin{tikzcd}
R \arrow[r, "\cdot c^{1/p}"] & R^{1/p} \arrow[r, "\rho"] & R \\
1 \arrow[r, mapsto] & c^{1/p} \arrow[r, mapsto] & 1.
\end{tikzcd}$$

Hence the relevant map into $R^{1/p}$ splits as a map of R-modules.

Definition 1.9 (F-split ring)**.** A ring R is called *F-split* if the natural inclusion

$$R \longrightarrow R^{1/p}$$

splits as a map of R-modules.

Remark 1.10. Equivalently, R is F-split if there exists an R-linear map

$$\varphi : R^{1/p} \to R$$

such that $\varphi(1^{1/p}) = 1$. Such a map is called a Frobenius splitting.

Regular Rings are Strongly F-Regular

Now assume R is regular and F-finite. Let

$$c \in \mathfrak{m}, \qquad c \neq 0.$$

Since

$$\bigcap_{\ell \geq 1} \mathfrak{m}^\ell = 0$$

by the Krull Intersection Theorem, we have

$$c \notin \mathfrak{m}^\ell$$

for all sufficiently large ℓ. If

$$\mathfrak{m} = (x_1, \ldots, x_d),$$

then for $\ell \gg 0$,

$$c \notin (x_1^\ell, \dots, x_d^\ell).$$

In particular, for $e \gg 0$,

$$c \notin (x_1^{p^e}, \dots, x_d^{p^e}).$$

Claim 1.11. *For $e \gg 0$, the class of c^{1/p^e} in $R^{1/p^e}/\mathfrak{m}R^{1/p^e}$ is nonzero.*

Proof. Under the p^e-th power identification $R^{1/p^e} \cong R$, the submodule $\mathfrak{m}R^{1/p^e}$ corresponds to the ideal

$$(x_1^{p^e}, \dots, x_d^{p^e}) \subseteq R.$$

Thus $c^{1/p^e} \in \mathfrak{m}R^{1/p^e}$ would imply

$$c \in (x_1^{p^e}, \dots, x_d^{p^e}),$$

which is false for $e \gg 0$. Therefore the image of c^{1/p^e} in $R^{1/p^e}/\mathfrak{m}R^{1/p^e}$ is nonzero. ☐

Since

$$\mathfrak{m}R^{1/p^e} \subseteq R^{1/p^e} \quad \longleftrightarrow \quad (x_1^{p^e}, \dots, x_d^{p^e}) \subseteq R,$$

the element c^{1/p^e} is part of a basis for the vector space

$$R^{1/p^e}/\mathfrak{m}R^{1/p^e}$$

over the residue field $R/\mathfrak{m}$. By Nakayama's Lemma, using the F-finite assumption, c^{1/p^e} is part of a basis for R^{1/p^e} as an R-module. Hence there exists an R-linear projection

$$\rho : R^{1/p^e} \longrightarrow R$$

which sends

$$c^{1/p^e} \longmapsto 1.$$

Consequently, the map

$$R \xrightarrow{\cdot c^{1/p^e}} R^{1/p^e}$$

splits.

Definition 1.12 (Strongly F-regular ring)**.** Let R be F-finite. The ring R is called *strongly F-regular*, abbreviated *SFR*, if for every $c \neq 0$, there exists some $e > 0$ such that the map

$$R \xrightarrow{\cdot c^{1/p^e}} R^{1/p^e}$$

splits as a map of R-modules.

Lemma 1.13. *If R is F-finite and regular, then R is strongly F-regular.*

Proof. The preceding argument proves the statement. For every nonzero $c \in R$, one chooses $e \gg 0$ so that

$$c \notin (x_1^{p^e}, \dots, x_d^{p^e}).$$

Then c^{1/p^e} survives modulo $\mathfrak{m}R^{1/p^e}$, can be extended to a basis of R^{1/p^e}, and admits an R-linear projection to R sending it to 1. This gives the desired splitting. ☐

Remark 1.14. Strong F-regularity is a characteristic-p analogue of mild singularities. In many contexts it corresponds, after reduction mod p, to Kawamata log terminal singularities in birational geometry.

Regular Quotients and Examples

Remark 1.15. If one quotients a regular local ring $(A, \mathfrak{m})$ by an ideal

$$0 \neq I \subseteq \mathfrak{m}^2,$$

then the quotient A/I is never regular.

Remark 1.16. Indeed, if $I \subseteq \mathfrak{m}^2$, then the embedding dimension of A/I remains the same as that of A, while the dimension drops whenever $I \neq 0$. Thus the equality

$$\dim(A/I) = \operatorname{edim}(A/I)$$

required for regularity fails.

Example 1.17 (Basic examples)**.** The following examples illustrate the gap between regularity, strong F-regularity, and F-splitting.

(1) The ring

$$\mathbb{F}_p[x, y, z]_{\mathfrak{m}}/(x^2 - yz)$$

is not regular, but it is strongly F-regular. Hence it is F-split.

Remark 1.18. This is an A_1-singularity. There is a standard isomorphism with the second Veronese subring:

$$\frac{k[x,y,z]}{(x^2-yz)} \cong k[ab,a^2,b^2] \subseteq_{\text{split}} k[a,b].$$

(2) For $p \neq 3$, the ring

$$\frac{\mathbb{F}_p[x,y,z]}{(x^3+y^3+z^3)}$$

is never strongly F-regular. Its F-splitting behavior depends on p modulo 3:

  (i) it is F-split if

$$p \equiv 1 \pmod 3;$$

  (ii) it is not F-split if

$$p \equiv 2 \pmod 3.$$

(3) The ring

$$\mathbb{F}_p[x,y,z]_{\mathfrak{m}}/(x^4+y^4+z^4)$$

is never F-split, even for $p \neq 2$.

Remark 1.19. The cubic example reflects the arithmetic of the Fermat cubic. The congruence condition on p controls whether the relevant Frobenius action admits a splitting. The quartic example shows that F-splitting can fail uniformly across all characteristics.

How to Check These Properties

To check F-splitting, one strategy is to explicitly exhibit a splitting: find an R-linear map

$$\varphi : R^{1/p} \longrightarrow R$$

such that

$$\varphi(1) = 1.$$

Another useful principle is descent along a split inclusion. If

$$R \hookrightarrow S$$

splits as a map of R-modules and S is F-split, then R is F-split.

Proposition 1.20 (Descent of F-splitting along direct summands)**.** *Suppose $R \hookrightarrow S$ is an inclusion of rings that splits as an R-module map. If S is F-split, then R is F-split.*

Proof. Let

$$\pi : S \to R$$

be an R-linear retraction of the inclusion $R \hookrightarrow S$. Since S is F-split, there exists an S-linear map

$$\Phi : S^{1/p} \to S$$

such that $\Phi(1) = 1$. Restricting along $R^{1/p} \to S^{1/p}$ and then composing with π, we obtain

$$R^{1/p} \longrightarrow S^{1/p} \xrightarrow{\Phi} S \xrightarrow{\pi} R.$$

This composite sends 1 to 1, so it gives an R-linear splitting of

$$R \to R^{1/p}.$$

Therefore R is F-split. ☐

Logical Implications

For F-finite rings in characteristic $p > 0$, the lecture establishes the following implications:

$$\text{regular} \longrightarrow \text{strongly } F\text{-regular} \longrightarrow F\text{-split}.$$

The examples above show that the converses do not hold in general.

2 Lecture-2 May 19, 2026

Throughout this lecture, let R be a domain of finite type over a field k of characteristic 0. Frequently R is assumed to be local, say $(R, \mathfrak{m})$. We write

$$X = \operatorname{Spec} R.$$

What Is Resolution of Singularities?

Definition 2.1 (Resolution of singularities)**.** A *resolution of singularities* of $X = \operatorname{Spec} R$ is a proper, usually projective, birational morphism

$$\pi : Y \longrightarrow \operatorname{Spec} R$$

such that Y is non-singular.

In the affine setting, a typical projective birational morphism is obtained as a blowup. Namely, if $I \subseteq R$ is an ideal, then the blowup of $\operatorname{Spec} R$ along I is

$$Y = \operatorname{Proj} S, \qquad S = R \oplus It \oplus I^2t^2 \oplus \cdots .$$

Here $S = \bigoplus_{n \geq 0} I^n t^n$ is the Rees algebra of I. The adjective *birational* means that π is an isomorphism over a dense open subset $U \subseteq \operatorname{Spec} R$. Equivalently, Y and $\operatorname{Spec} R$ have the same function field:

$$k(Y) = \operatorname{Frac}(R).$$

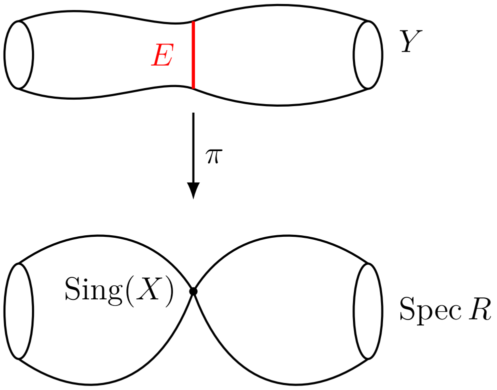

Figure 1. A proper birational morphism resolving the singularity of $\operatorname{Spec} R$. The locus $E \subseteq Y$ is the exceptional set.

Definition 2.2 (Exceptional set)**.** Let $\pi : Y \to X$ be a resolution of singularities. The *exceptional set* is the closed subset

$$E = \operatorname{Exc}(\pi) \subseteq Y$$

where π is not an isomorphism.

For resolutions in characteristic 0, one may arrange several additional properties.

Proposition 2.3 (Standard refinements of resolution)**.** *Let R be as above and let $X = \operatorname{Spec} R$. After replacing a resolution by a sufficiently refined one, we may assume the following.*

(1) *The morphism*

$$\pi : Y \to X$$

is an isomorphism over the non-singular locus of X, that is, over $X \setminus \operatorname{Sing}(X)$.

(2) *The exceptional set*

$$E = \operatorname{Exc}(\pi)$$

is a divisor with simple normal crossings. In other words, every irreducible component of E is non-singular, and locally on Y, the divisor E is defined by a partial product of a regular system of parameters.

(3) *More generally, for any ideal $J \subseteq R$, we may arrange that*

$$V(J\mathcal{O}_Y) \cup E$$

has simple normal crossings.

Definition 2.4 (Simple normal crossings)**.** Let Y be a non-singular variety. A divisor $D \subseteq Y$ has *simple normal crossings*, abbreviated SNC, if for every point $y \in Y$, there is a regular system of parameters

$$x_1, \dots, x_d$$

in the regular local ring $\mathcal{O}_{Y,y}$ such that D is locally defined by an equation of the form

$$x_1 x_2 \cdots x_r = 0$$

for some $0 \leq r \leq d$.

Thus, locally at a point $y \in Y$, one can write

$$\mathcal{O}_{Y,y} \cong (R', \mathfrak{m}')$$

where

$$\mathfrak{m}' = (x_1, \dots, x_d),$$

and the exceptional divisor has normal-crossing form, for example

$$E = V(x_1 x_2 \cdots x_r).$$

Remark 2.5. The condition that E be simple normal crossings is crucial because it reduces many local computations on the resolution to computations with monomials in a regular local ring. This is why resolution of singularities is often paired with log-resolution statements: after further blowups, the exceptional divisor and the total transform of a chosen closed subscheme can be made simultaneously monomial in suitable local coordinates.

Example: Blowing Up a Fermat Surface Singularity

Example 2.6. Let

$$R = \frac{k[x,y,z]}{(x^4+y^4+z^4)}$$

and let

$$\mathfrak{m} = (x,y,z) \subseteq R.$$

We blow up $X = \operatorname{Spec} R$ at the maximal ideal $\mathfrak{m}$. The blowup is

$$Y = \operatorname{Proj}\Big(R \oplus \mathfrak{m}t \oplus \mathfrak{m}^2t^2 \oplus \cdots\Big).$$

We analyze Y on the standard affine charts.

The x-chart

On the chart where x is the chosen generator of the blowup, introduce coordinates

$$u = \frac{y}{x}, \qquad v = \frac{z}{x}.$$

The corresponding affine chart of the blowup is described by

$$k[x,u,v]\Big/(x^4 + x^4u^4 + x^4v^4).$$

Equivalently,

$$k[x,u,v]\Big/(x^4(1+u^4+v^4)).$$

The strict transform is obtained by removing the exceptional factor x^4. Hence the strict transform on this chart has coordinate ring

$$k[x,u,v]\Big/(1+u^4+v^4),$$

that is,

$$k\left[x, \frac{y}{x}, \frac{z}{x}\right]\Big/\left(1+\left(\frac{y}{x}\right)^4+\left(\frac{z}{x}\right)^4\right).$$

Let

$$F_1 = 1+u^4+v^4 = 1+\left(\frac{y}{x}\right)^4+\left(\frac{z}{x}\right)^4.$$

The exceptional divisor in this chart is

$$E = V(x).$$

To check non-singularity of this chart, compute

$$\frac{\partial F_1}{\partial u} = 4u^3, \qquad \frac{\partial F_1}{\partial v} = 4v^3, \qquad \frac{\partial F_1}{\partial x} = 0.$$

Since $\mathrm{char}(k) = 0$, the equations

$$F_1 = \frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial v} = 0$$

would force $u = v = 0$, but then $F_1 = 1 \neq 0$. Hence by Jacobian Criteria there are no singular points on this chart:

$$V\left(F_1, \frac{\partial F_1}{\partial u}, \frac{\partial F_1}{\partial v}\right) = \varnothing.$$

The y-chart

On the chart where y is the chosen generator, set

$$a = \frac{x}{y}, \qquad b = \frac{z}{y}.$$

The strict transform is given by

$$k[a, y, b] \Big/ (a^4 + 1 + b^4),$$

that is,

$$k\left[\frac{x}{y}, y, \frac{z}{y}\right] \Big/ \left(\left(\frac{x}{y}\right)^4 + 1 + \left(\frac{z}{y}\right)^4\right).$$

The same Jacobian computation shows that this chart is non-singular.

The z-chart

On the chart where z is the chosen generator, set

$$c = \frac{x}{z}, \qquad d = \frac{y}{z}.$$

The strict transform is given by

$$k[c, d, z] \Big/ (c^4 + d^4 + 1),$$

that is,

$$k\left[\frac{x}{z}, \frac{y}{z}, z\right] \Big/ \left(\left(\frac{x}{z}\right)^4 + \left(\frac{y}{z}\right)^4 + 1\right).$$

Again, by symmetry, this chart is non-singular. Therefore, the blowup at $\mathfrak{m}$ resolves the singularity in this example.

Remark 2.7. The displayed chart rings above describe the strict transform. The total transform contains a common exceptional factor, such as x^4 on the x-chart. The strict transform is the geometrically relevant component for the resolution, obtained by dividing out this exceptional factor.

Cohomology and Rational Singularities

The goal is to relate R, its normalization, and related extensions such as $R^{1/p}$ or R^{1/p^e} in positive characteristic, to the geometry of a resolution

$$\pi : Y \to \operatorname{Spec} R.$$

A first approximation is to take global sections:

$$\Gamma(Y, \mathcal{O}_Y).$$

For a proper birational morphism from a normal variety Y, one obtains

$$\Gamma(Y, \mathcal{O}_Y) = R^N,$$

where R^N denotes the normalization of R inside its fraction field. However, $\Gamma(Y, \mathcal{O}_Y)$ only records degree-zero cohomological information. Instead, one studies the derived global sections

$$R\Gamma(Y, \mathcal{O}_Y).$$

This is a complex whose cohomology groups are

$$H^i(Y, \mathcal{O}_Y) \qquad (i \geq 0).$$

It can be viewed as a differential graded algebra, and in the setting of rational singularities it is independent, up to quasi-isomorphism, of the chosen resolution of singularities.

Remark 2.8. The derived object $R\Gamma(Y, \mathcal{O}_Y)$ is better behaved than ordinary global sections because it remembers all higher cohomology. Rational singularities are precisely those singularities for which this higher cohomological information contributes nothing beyond the original ring R.

Computing Derived Global Sections by a Čech Complex

Let

$$Y = \bigcup_i U_i$$

be an affine open cover. Then $R\Gamma(Y, \mathcal{O}_Y)$ is represented by the Čech complex

$$\check{C}^\bullet(\{U_i\}, \mathcal{O}_Y):$$

$$\prod_i \Gamma(U_i, \mathcal{O}_Y) \longrightarrow \prod_{i<j} \Gamma(U_i \cap U_j, \mathcal{O}_Y) \longrightarrow \prod_{i<j<\ell} \Gamma(U_i \cap U_j \cap U_\ell, \mathcal{O}_Y) \longrightarrow \cdots.$$

Thus there is a quasi-isomorphism

$$R\Gamma(Y, \mathcal{O}_Y) \simeq \check{C}^\bullet(\{U_i\}, \mathcal{O}_Y).$$

For the previous blowup example, the first terms of the Čech complex are of the form

$$\left(k\left[x, \frac{y}{x}, \frac{z}{x}\right]\Big/(\cdots)\right) \oplus \left(k\left[\frac{x}{y}, y, \frac{z}{y}\right]\Big/(\cdots)\right) \oplus \left(k\left[\frac{x}{z}, \frac{y}{z}, z\right]\Big/(\cdots)\right) \longrightarrow \prod_{i<j} \Gamma(U_i \cap U_j, \mathcal{O}_Y) \longrightarrow \cdots.$$

For example, one of the pairwise intersections is represented by a ring of the form

$$k\left[x, \frac{y}{x}, \frac{z}{x}, \frac{x}{y}\right]\Big/(\cdots),$$

where the omitted ideal is obtained by imposing the strict-transform equation and the compatibility relations among the displayed coordinates.

Definition 2.9 (Rational singularities)**.** Let R be a domain essentially of finite type over a field of characteristic 0, and let

$$\pi : Y \to X = \operatorname{Spec} R$$

be a resolution of singularities. We say that R, or X, has *rational singularities* if the natural map

$$R \longrightarrow R\Gamma(Y, \mathcal{O}_Y)$$

is a quasi-isomorphism.

Equivalently, R has rational singularities if

$$R \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y) = R^N$$

and

$$H^i(Y, \mathcal{O}_Y) = 0 \qquad \text{for all } i > 0.$$

Example 2.10. One can prove that

$$R = \frac{k[x,y,z]}{(x^4 + y^4 + z^4)}$$

does not have rational singularities by showing that, for the resolution Y obtained above,

$$H^1(Y, \mathcal{O}_Y) \neq 0.$$

Open Complements and Local Cohomology

Let $J \subseteq R$ be an ideal generated by

$$J = (f_1, \dots, f_n).$$

Set

$$U = \operatorname{Spec} R \setminus V(J).$$

The open subset U is covered by the standard affine opens

$$U_i = D(f_i) = \operatorname{Spec} R[f_i^{-1}].$$

The derived global sections $R\Gamma(U, \mathcal{O}_U)$ are computed by the corresponding Čech complex:

$$R \longrightarrow \prod_i R[f_i^{-1}] \longrightarrow \prod_{i<j} R[(f_i f_j)^{-1}] \longrightarrow \cdots .$$

In the context of a resolution Y, one similarly obtains complexes such as

$$0 \longrightarrow R \longrightarrow R\Gamma(Y, \mathcal{O}_Y)[f_1^{-1}] \longrightarrow \cdots ,$$

which represent Čech-type constructions associated to local cohomology. In particular, the local cohomology complex $R\Gamma_J(R)$ is modeled by the cone of

$$R \longrightarrow R\Gamma(U, \mathcal{O}_U),$$

up to the standard cohomological shift conventions.

Remark 2.11. The relationship between Čech complexes and local cohomology is as follows. If $J = (f_1, \dots, f_n)$, then the ordinary Čech complex on $f_1, \dots, f_n$ computes $R\Gamma_J(R)$. Equivalently, the complement $\operatorname{Spec} R \setminus V(J)$ is covered by the opens $D(f_i)$, and local cohomology measures the difference between global sections on $\operatorname{Spec} R$ and derived global sections on this open complement.

Why Are Rational Singularities Good?

Normality

Proposition 2.12. *Rational singularities are normal.*

Proof. If R has rational singularities, then by definition the map

$$R \longrightarrow R\Gamma(Y, \mathcal{O}_Y)$$

is a quasi-isomorphism. Taking zeroth cohomology gives an isomorphism

$$R \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y).$$

For a resolution $Y \to \operatorname{Spec} R$, the ring of global sections is the normalization:

$$\Gamma(Y, \mathcal{O}_Y) \cong R^N.$$

Therefore

$$R \cong R^N,$$

so R is normal. □

Cohen–Macaulayness

Proposition 2.13. *Rational singularities are Cohen–Macaulay.*

Proof. Let $(R, \mathfrak{m})$ be local of dimension d, and suppose R has rational singularities. Then

$$R \simeq R\Gamma(Y, \mathcal{O}_Y)$$

in the derived category. Hence local cohomology gives

$$H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}\Big(R\Gamma(Y, \mathcal{O}_Y)\Big).$$

Using local duality together with the vanishing theorems available for resolutions in characteristic 0, one obtains

$$H^i_{\mathfrak{m}}(R) = 0 \qquad \text{for all } i < \dim R.$$

Therefore

$$\operatorname{depth} R = \dim R,$$

so R is Cohen–Macaulay. □

Remark 2.14. The proof that rational singularities are Cohen–Macaulay is not merely a formal consequence of the definition. It uses deep vanishing results, such as Grauert–Riemenschneider vanishing, together with local duality. The guiding principle is that the higher direct images of $\mathcal{O}_Y$ vanish for rational singularities, and the dualizing sheaf on the smooth resolution has strong vanishing properties.

Grauert–Riemenschneider Vanishing

Theorem 2.15 (Grauert–Riemenschneider vanishing)**.** *Let*

$$\pi : Y \to \operatorname{Spec} R$$

be a resolution of singularities, and let ω_Y be the canonical sheaf of the non-singular variety Y. Then

$$R\Gamma(Y, \omega_Y)$$

is a Cohen–Macaulay complex. Consequently,

$$H^i(Y, \omega_Y) = 0$$

in the relevant cohomological range below the dimension of R.

More precisely, the Cohen–Macaulayness of the complex means that its local cohomology is concentrated in the expected top degree. In the local case, this is the form compatible with local duality and the Cohen–Macaulayness of rings with rational singularities.

Splitting Characterization of Rational Singularities

Theorem 2.16 (Kovács–Bhatt splitting criterion)**.** *Let*

$$X = \operatorname{Spec} R$$

be a variety, or more generally a scheme in the appropriate setting. Then X has rational singularities if and only if for all projective surjective morphisms

$$\pi : Y \to X,$$

the natural map

$$\mathcal{O}_X \longrightarrow R\pi_*\mathcal{O}_Y$$

splits in the derived category.

The lecture diagram can be rendered schematically as follows:

$$\begin{tikzcd}
\mathrm{RoS} \arrow[dr, "\text{projective}"] & Y \arrow[d] & Z \arrow[l, "\text{finite}"'] \arrow[d] \\
& \operatorname{Spec} R & Z' \arrow[l]
\end{tikzcd}$$

Remark 2.17. The splitting condition says that $\mathcal{O}_X$ is a direct summand of $R\pi_*\mathcal{O}_Y$ in the derived category. This is stronger than merely asking for a splitting on ordinary global sections. The derived splitting controls all higher cohomological obstructions at once.

Reduction to Positive Characteristic

Theorem 2.18 (Smith–Hara–Mehta–Srinivas)**.** *Let $R_{\mathbb{Z}}$ be a ring of finite type over $\mathbb{Z}$, and set*

$$R_{\mathbb{Q}} = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Assume that $R_{\mathbb{Q}}$ is Gorenstein. Then $R_{\mathbb{Q}}$ has rational singularities if and only if, for all sufficiently large primes p, the characteristic p reduction

$$R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$$

has strongly F-regular singularities at each localization.

Remark 2.19. This theorem is one of the central bridges between characteristic 0 singularity theory and F-singularity theory. Rational singularities in characteristic 0 correspond, after reduction modulo $p \gg 0$, to strong F-regularity in characteristic p, at least under the stated Gorenstein hypothesis.

3 Lecture-3 May 20, 2026

Let $(A, \mathfrak{m})$ be an F-finite regular local ring, and let

$$0 \neq f \in \mathfrak{m}.$$

The goal is to measure the singularities of the hypersurface

$$R = A/(f).$$

For each integer $e \geq 1$, consider the finite A-algebra A^{1/p^e}. An element of A^{1/p^e} is formally written as a p^e-th root of an element of A. The basic maps under consideration are

$$A \longrightarrow A^{1/p^e}, \qquad 1 \longmapsto f^{a/p^e}.$$

Equivalently, one studies the A-linear map determined by multiplication by f^{a/p^e} after applying the e-fold frobenius-root extension.

Definition 3.1 (F-pure threshold)**.** With notation as above, the *F-pure threshold* of f is

$$\operatorname{fpt}(f) = \sup \left\{ \frac{a}{p^e} \;\middle|\; A \xrightarrow{1 \mapsto f^{a/p^e}} A^{1/p^e} \text{ splits as an } A\text{-module map} \right\}$$

Remark 3.2. Here "splits" means that there exists an A-linear map

$$\psi : A^{1/p^e} \longrightarrow A$$

such that

$$\psi(f^{a/p^e}) = 1.$$

Thus the condition is equivalent to saying that f^{a/p^e} generates a direct summand copy of A inside A^{1/p^e}.

Remark 3.3 (Well-definedness)**.** Nothing pathological occurs when the same rational number is written with different p-power denominators. for example,

$$\frac{a}{p^e} = \frac{ap}{p^{e+1}}.$$

The compatibility is encoded in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "1 \mapsto f^{a/p^e}"] \arrow[d, equal] & A^{1/p^e} \arrow[r, "\text{splitting}"] \arrow[d, hook, "\text{split}"'] & A \\
A \arrow[r, "1 \mapsto f^{ap/p^{e+1}}"'] & A^{1/p^{e+1}} \arrow[ur, dashed] &
\end{tikzcd}$$

Indeed, the natural inclusion $A^{1/p^e} \hookrightarrow A^{1/p^{e+1}}$ sends

$$f^{a/p^e} \longmapsto f^{ap/p^{e+1}}.$$

Therefore the splitting condition is stable under replacing a/p^e by ap^n/p^{e+n}.

Fact 3.4. One has

$$\boxed{0 < \operatorname{fpt}(f) \leq 1.}$$

Roughly speaking, a larger value of $\operatorname{fpt}(f)$ means that the hypersurface $A/(f)$ is less singular.

Remark 3.5. The upper bound $\operatorname{fpt}(f) \leq 1$ is special to the principal ideal case. for a hypersurface, the threshold reaches its maximal value precisely when the hypersurface has the expected F-singularity property, namely F-purity.

Example 3.6. The following examples are standard benchmarks:

$$\operatorname{fpt}\Big(xy \in \mathbb{F}_p[x,y]\Big) = 1,$$
$$\operatorname{fpt}\Big(x^2 - yz \in \mathbb{F}_p[x,y,z]\Big) = 1,$$

and, for the cusp,

$$\operatorname{fpt}\Big(y^2 - x^3 \in \mathbb{F}_p[x,y]\Big) = \begin{cases} \dfrac{5}{6}, & \text{if } p \equiv 1 \pmod 6, \\[2ex] \dfrac{1}{2}, & \text{if } p = 2, \\[2ex] \dfrac{2}{3}, & \text{if } p = 3, \\[2ex] \dfrac{5}{6} - \dfrac{1}{6p}, & \text{if } p \equiv 5 \pmod 6. \end{cases}$$

Lemma 3.7. *Let $R = A/(f)$. Then R is F-split if and only if*

$$\operatorname{fpt}(f) = 1.$$

Proof. We prove both directions carefully. First suppose that $\operatorname{fpt}(f) = 1$. Since $\operatorname{fpt}(f) \leq 1$, this means that the splitting condition holds arbitrarily close to 1. In particular, the map

$$A \longrightarrow A^{1/p}, \qquad 1 \longmapsto f^{(p-1)/p}$$

splits. Hence there exists an A-linear map

$$\psi : A^{1/p} \longrightarrow A$$

such that

$$\psi\Big(f^{(p-1)/p}\Big) = 1.$$

Consider the composite

$$\begin{tikzcd}
A \arrow[r, hook] & A^{1/p} \arrow[r, "\cdot f^{(p-1)/p}"] & A^{1/p} \arrow[r, "\psi"] & A \\
1 \arrow[r, mapsto] & 1^{1/p} \arrow[r, mapsto] & f^{(p-1)/p} \arrow[r, mapsto] & 1.
\end{tikzcd}$$

Let

$$\Phi = \psi \circ \Big(\cdot f^{(p-1)/p}\Big) : A^{1/p} \longrightarrow A.$$

Then Φ is A-linear and satisfies

$$\Phi(1^{1/p}) = 1.$$

Moreover,

$$\begin{aligned}
\Phi\Big((f^{1/p})A^{1/p}\Big) &= \psi\Big(f^{(p-1)/p}(f^{1/p})A^{1/p}\Big) \\
&= \psi\Big(fA^{1/p}\Big) \\
&\subseteq f\psi(A^{1/p}) \\
&\subseteq (f).
\end{aligned}$$

Therefore Φ descends modulo (f) to an R-linear map

$$\overline{\Phi} : R^{1/p} \longrightarrow R.$$

The descent is represented by the commutative diagram

$$\begin{tikzcd}
(f^{1/p})A^{1/p} \arrow[r] \arrow[d, hook] & (f) \arrow[d, hook] \\
A^{1/p} \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\
A^{1/p}/(f^{1/p})A^{1/p} \arrow[r, "\overline{\Phi}"] \arrow[d] & A/(f) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

Since

$$A^{1/p}/(f^{1/p})A^{1/p} \cong R^{1/p}$$

and

$$\overline{\Phi}(1_R^{1/p}) = 1_R,$$

the Frobenius map

$$R \longrightarrow R^{1/p}$$

splits. Thus $R = A/(f)$ is F-split. Conversely, suppose that $R = A/(f)$ is F-split. Then there exists an R-linear map

$$\varphi : R^{1/p} \longrightarrow R$$

such that

$$\varphi(1_R^{1/p}) = 1_R.$$

By the usual Fedder-type lifting criterion for a hypersurface in a regular local ring, this is equivalent to the existence of an A-linear map

$$\psi : A^{1/p} \longrightarrow A$$

such that

$$\psi\Big(f^{(p-1)/p}\Big) = 1.$$

Hence the map

$$A \longrightarrow A^{1/p}, \qquad 1 \longmapsto f^{(p-1)/p}$$

splits. Therefore

$$\frac{p-1}{p} \leq \operatorname{fpt}(f).$$

Iterating the same argument for the e-fold frobenius gives splittings

$$A \longrightarrow A^{1/p^e}, \qquad 1 \longmapsto f^{(p^e-1)/p^e}$$

for all $e \geq 1$. Consequently

$$\frac{p^e-1}{p^e} \leq \operatorname{fpt}(f) \qquad \text{for all } e \geq 1.$$

Letting $e \to \infty$, we obtain

$$1 \leq \operatorname{fpt}(f).$$

Since always $\operatorname{fpt}(f) \leq 1$, it follows that

$$\operatorname{fpt}(f) = 1.$$

□

The perfect Closure and the Absolute Integral Closure

Remark 3.8 (Rodríguez–Villalobos)**.** Define

$$A_{\text{perf}} = \bigcup_{e>0} A^{1/p^e}$$

This ring is usually not Noetherian. for example, if

$$A = \mathbb{F}_p[[x]],$$

then the ideal

$$(x, x^{1/p}, x^{1/p^2}, \ldots) \subseteq A_{\text{perf}}$$

is not finitely generated. Using A_{perf}, the f-pure threshold can be rewritten as

$$\operatorname{fpt}(f) = \sup\left\{ \frac{a}{p^e} \;\middle|\; A \xrightarrow{1 \mapsto f^{a/p^e}} A_{\text{perf}} \text{ is pure} \right\}.$$

If $A = \widehat{A}$, then purity can be detected by splitting in this setting.

Now set

$$A^+ = \text{the integral closure of } A \text{ in } \overline{K(A)}$$

. Equivalently,

$$A^+ = \bigcup_{\substack{A \subseteq S \subseteq \overline{K(A)} \\ S \text{ finite over } A}} S$$

The ring A^+ is generally non-Noetherian. In these terms, one can write

$$\operatorname{fpt}(f) = \sup\left\{ t \in \mathbb{Q} \;\middle|\; A \xrightarrow{1 \mapsto f^t} A^+ \text{ is pure} \right\},$$

with the additional note that if $A = \widehat{A}$, then the relevant pure maps split.

Remark 3.9. The notation f^t for rational t should be understood after passing to a finite extension S of A in which the required root of f exists. Any two choices of a root representing f^t differ by a unit, so the resulting condition is well-defined. Thus the expression

$$A \xrightarrow{1 \mapsto f^t} A^+$$

packages all finite extensions containing the relevant fractional power of f.

Characteristic 0

Let $(A, \mathfrak{m})$ be a regular local ring essentially of finite type over a field k, where

$$\operatorname{char} k = 0,$$

and let

$$0 \neq f \in \mathfrak{m}.$$

Definition 3.10 (Log canonical threshold)**.** The *log canonical threshold* of f is

$$\operatorname{lct}(f) = \sup\left\{ t \in \mathbb{Q}_{\geq 0} \;\middle|\; \begin{array}{l} \forall S \supseteq A \text{ finite with } f^t \in S, \\ \text{for every resolution of singularities } Y \to \operatorname{Spec} S, \\ A \to S \xrightarrow{1 \mapsto f^t} \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \text{ splits} \end{array} \right\}.$$

Remark 3.11. Here “RoS” in the lecture notes means resolution of singularities. The object

$$\mathbf{R}\Gamma(Y, \mathcal{O}_Y)$$

is the derived global section complex of the structure sheaf. The splitting condition should be interpreted in the derived category, or equivalently after applying the appropriate derived Hom functor. In the hypersurface case, this threshold agrees with the usual birational-geometric log canonical threshold.

Example 3.12. The characteristic-zero analogues of the previous examples are

$$\begin{aligned}\mathrm{lct}\Big(xy \in \mathbb{Q}[x,y]\Big) &= 1,\\ \mathrm{lct}\Big(x^2 - yz \in \mathbb{Q}[x,y,z]\Big) &= 1,\\ \mathrm{lct}\Big(y^2 - x^3 \in \mathbb{C}[x,y]\Big) &= \frac{5}{6}.\end{aligned}$$

Fact 3.13. for $0 \neq f \in \mathfrak{m}$, one has:

(1)

$$0 < \mathrm{lct}(f) \leq 1.$$

(2)

$$\mathrm{lct}(f) = 1 \quad\Longleftrightarrow\quad A/(f) \text{ is log canonical}$$

Moreover, in the Gorenstein setting, rational singularities imply log canonical singularities:

$$A/(f) \text{ Gorenstein rational} \quad\Longrightarrow\quad A/(f) \text{ log canonical.}$$

Theorem 3.14 (Takagi–Watanabe; Smith; Hara; Mehta–Srinivas)**.** *Let*

$$A_{\mathbb{Z}} = \mathbb{Z}[x_1, \dots, x_n],$$

let

$$A_{\mathbb{Q}} = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q},$$

and, for a prime p, let

$$A_p = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$$

Let

$$0 \neq f \in (x_1, \dots, x_n)A_{\mathbb{Z}},$$

and let

$$f_p = \mathrm{im}(f) \in A_p$$

be its reduction modulo p. Then

$$\lim_{p\to\infty} \mathrm{fpt}(f_p \in A_p) = \mathrm{lct}(f \in A_{\mathbb{Q}}).$$

Remark 3.15. This theorem expresses the fundamental compatibility between f-singularities in large positive characteristic and birational singularities in characteristic zero. The log canonical threshold is recovered as the limiting value of f-pure thresholds of reductions modulo p.

Mixed Characteristic

Let$(A, \mathfrak{m})$ be a complete regular local ring of mixed characteristic $(0, p > 0)$. Thus

$$\operatorname{char} A = 0 \qquad \text{and} \qquad \operatorname{char}(A/\mathfrak{m}) = p > 0.$$

Example 3.16. Typical examples include

$$A = \mathbb{Z}_p[[x_2, \dots, x_n]], \qquad \mathfrak{m} = (p, x_2, \dots, x_n),$$

and

$$A = \mathbb{Z}_p[[x_2, \dots, x_n]]\left[p^{1/p}\right], \qquad \mathfrak{m} = (p^{1/p}, x_2, \dots, x_n).$$

Theorem 3.17 (Bhatt)**.** *Let $(A, \mathfrak{m})$ be a Noetherian complete local domain of mixed characteristic $(0, p)$. Let A^+ denote the absolute integral closure of A. Then the p-adic completion*

$$(A^+)^{\wedge_p}$$

is a perfectoid balanced big Cohen–Macaulay A-algebra.
Consequently, its $\mathfrak{m}$-adic completion, equivalently the $\mathfrak{m}$-adic completion of A^+ in this setting, may also be used as a balanced big Cohen–Macaulay A-algebra.

Remark 3.18. A big Cohen–Macaulay A-algebra B is an A-algebra on which every system of parameters of A is a regular sequence. Bhatt's theorem is one of the key inputs allowing one to formulate mixed-characteristic analogues of characteristic-p splitting thresholds.

Definition 3.19 (plus pure threshold)**.** Let $0 \neq f \in \mathfrak{m}$. The *plus pure threshold* of f is

$$\begin{aligned} \operatorname{ppt}(f) &= \sup\left\{ t \in \mathbb{Q}_{>0} \;\middle|\; A \xrightarrow{1 \mapsto f^t} (A^+)^{\wedge_p} \text{ is pure} \right\} \\ &= \sup\left\{ t \in \mathbb{Q}_{>0} \;\middle|\; A \xrightarrow{1 \mapsto f^t} (A^+)^{\wedge_p} \text{ splits} \right\} \\ &= \sup\left\{ t \in \mathbb{Q}_{>0} \;\middle|\; f^t \notin \mathfrak{m}(A^+)^{\wedge_p} \right\} \\ &= \sup\left\{ t \in \mathbb{Q}_{>0} \;\middle|\; f^t \notin \mathfrak{m}A^+ \right\} \\ &= \sup\left\{ t \in \mathbb{Q}_{>0} \;\middle|\; f^t \notin \mathfrak{m}S \quad \forall S \supseteq A \text{ finite with } f^t \in S \right\}. \end{aligned}$$

Remark 3.20. The equality between the splitting formulation and the condition

$$f^t \notin \mathfrak{m}(A^+)^{\wedge p}$$

is a mixed-characteristic analogue of the elementary local criterion for a map

$$A \to B, \qquad 1 \mapsto b,$$

to split when B behaves like a big Cohen–Macaulay algebra.
In a local situation, the obstruction to splitting is measured by whether b lands inside the extension of the maximal ideal.

Fact 3.21. for $0 \neq f \in A$, the thresholds satisfy

$$\mathrm{fpt}(\overline{f} \in A/p) \leq \mathrm{ppt}(f \in A) \leq \mathrm{lct}(f \in A_{\mathbb{Q}}).$$

Example 3.22. The notes record the following mixed-characteristic comparison:

$$\mathrm{ppt}\Big(p^2 - x^3 \in A \text{ of mixed characteristic } (0,p)\Big) = \mathrm{fpt}\Big(y^2 - x^3 \in \mathbb{F}_p[x,y]\Big).$$

They also record the expression

$$\begin{aligned} \mathrm{ppt}(8 + y^3 + z^3) &\leq 1, \\ &< \frac{3}{4}. \end{aligned}$$

Question 3.23 (Tucker's Bane)**.** Compute

$$\mathrm{ppt}\Big(x^3 + 27 \in \mathbb{Z}_3[x]\Big).$$

4 Lecture-4 May 21, 2026

Definition 4.1 (Big Cohen–Macaulay algebras)**.** Let $(R, \mathfrak{m})$ be an excellent local domain. In the present notes, "excellent" means, for example, complete or essentially of finite type over a field k, over $\mathbb{Z}$, over $\mathbb{Z}_{(p)}$, and similar standard excellent base rings.
An R-algebra B is called **weakly balanced big Cohen–Macaulay** if every system of parameters on R is a regular sequence on B. It is called **balanced big Cohen–Macaulay** if, in addition,

$$\mathfrak{m}B \neq B.$$

We abbreviate this by saying that B is a **BCM algebra**.

Remark 4.2. The adjective "balanced" means that the regular sequence condition is required for every system of parameters, not merely for one chosen system of parameters. The condition $\mathfrak{m}B \neq B$ rules out the degenerate case in which the parameter sequence is vacuously regular because the algebra has collapsed.

Theorem 4.3 (Hochster–Huneke, André, Gabber)**.** *Big Cohen–Macaulay algebras exist.*

Theorem 4.4. *Let $(R, \mathfrak{m})$ be an excellent regular local ring, and let B be a BCM R-algebra. Then B is flat as an R-module.*

Remark 4.5. For a regular local ring, every system of parameters is a regular system of parameters. Thus the BCM condition forces the homological behavior of B over R to be as good as possible. The theorem is a big Cohen–Macaulay analogue of the local criterion for flatness.

BCM-Regularity Along a Fixed BCM Algebra

Definition 4.6 (Weak BCM-regularity along a BCM algebra)**.** Let B be a BCM R-algebra. We say that R is **weakly BCM-regular along** B if the structure map

$$R \longrightarrow B$$

is pure. If R is complete, this is equivalent to saying that $R \to B$ splits as a map of R-modules.

Remark 4.7. A map $R \to B$ is pure if, for every R-module M, the induced map

$$M \longrightarrow M \otimes_R B$$

is injective. When R is complete local, purity of certain module-finite or sufficiently controlled maps can be tested by splitting criteria; this is the sense in which purity and splitting are being identified in the lecture notes.

Theorem 4.8 (Deformation statement for BCM-regularity)**.** *Let $(R, \mathfrak{m})$ be an excellent local domain. Assume that R is Gorenstein, and let $0 \neq f \in \mathfrak{m}$. Let B be a BCM*

> *R-algebra. Suppose that $R/(f)$ is BCM-regular along $B/(f)$. Then R is BCM-regular.*

Proof. In the notation below, put $\dim R = d$. The proof is phrased in terms of local cohomology. Let $\eta \in H^d_{\mathfrak{m}}(R)$. To prove injectivity of

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B),$$

let $\eta \in H^d_{\mathfrak{m}}(R)$ be an element whose image in $H^d_{\mathfrak{m}}(B)$ is zero. Since $H^d_{\mathfrak{m}}(R)$ is Artinian and $f \in \mathfrak{m}$, there exists $n \gg 0$ such that $f^n\eta = 0$. Choose n minimal with this property. Replacing η by $f^{n-1}\eta$, it is enough to treat the case

$$f\eta = 0.$$

The relevant portion of the long exact sequence in local cohomology associated to

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0$$

is

$$H^{d-1}_{\mathfrak{m}}(R/(f)) \xrightarrow{\gamma} H^d_{\mathfrak{m}}(R) \xrightarrow{f} H^d_{\mathfrak{m}}(R).$$

It suffices to show the required injectivity in the following diagram. The corresponding sequence for B and B/fB fits into the commutative diagram

$$\begin{tikzcd}
H^{d-1}_{\mathfrak{m}}(R) \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/(f)) \arrow[r, "\gamma"] \arrow[d] & H^d_{\mathfrak{m}}(R) \arrow[r, "f"] \arrow[d] & H^d_{\mathfrak{m}}(R) \arrow[d] \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(B/fB) \arrow[r] & H^d_{\mathfrak{m}}(B) \arrow[r, "f"] & H^d_{\mathfrak{m}}(B).
\end{tikzcd}$$

The lower-left zero reflects the Cohen–Macaulay behavior forced by the BCM condition since $\dim(R/f) = d-1$. Thus, to prove that the image of η in $H^d_{\mathfrak{m}}(B)$ is nonzero whenever η is nonzero, it is enough to prove injectivity at the indicated stage.

Since $f\eta = 0$, exactness gives an element

$$\xi \in H^{d-1}_{\mathfrak{m}}(R/(f))$$

such that

$$\gamma(\xi) = \eta.$$

The hypothesis that $R/(f)$ is BCM-regular along $B/(f)$ gives the required injectivity after mapping to $H^{d-1}_{\mathfrak{m}}(B/fB)$. Therefore the image of ξ does not vanish unless ξ already vanishes in the relevant quotient. By commutativity of the diagram, this forces the image of η in $H^d_{\mathfrak{m}}(B)$ to be nonzero.

It remains to explain why this injectivity criterion is enough. Because R is Gorenstein, the top local cohomology module $H^d_{\mathfrak{m}}(R)$ is the injective hull of the residue field. Consequently, purity or splitting of $R \to B$ can be tested on the induced map on top local cohomology, see remark below. This proves BCM-regularity of R. □

Remark 4.9. If R is a complete Gorenstein ring, and $R \to S$ is a ring map, then

$$\begin{aligned}
& R \to S \quad \text{is split} \\
\iff & \operatorname{Hom}_R(S,R) \to \operatorname{Hom}_R(R,R) \quad \text{is surjective} \\
\iff & \operatorname{Hom}_R(S,\operatorname{Hom}_R(E,E)) \to \operatorname{Hom}_R(R,\operatorname{Hom}_R(E,E)) \quad \text{is surjective} \\
\iff & \operatorname{Hom}_R(S\otimes_R E,E) \to \operatorname{Hom}_R(R\otimes_R E,E) \quad \text{is surjective} \\
\iff & R\otimes_R E \to S\otimes_R E \quad \text{is injective} \\
\iff & R\otimes_R H^{\dim R}_{\mathfrak{m}}(R) \to S\otimes_R H^{\dim R}_{\mathfrak{m}}(R) \quad \text{is injective} \\
\iff & R\otimes_R H^{\dim R}_{\mathfrak{m}}(R) \to S\otimes_R H^{\dim R}_{\mathfrak{m}}(R) \quad \text{is injective} \\
\iff & H^{\dim R}_{\mathfrak{m}}(R) \to H^{\dim R}_{\mathfrak{m}}(S) \quad \text{is injective}
\end{aligned}$$

Remark 4.10. The displayed diagram is obtained from the long exact local cohomology sequences associated to multiplication by f on R and on B. The vertical maps are induced by the structural map $R \to B$.

Remark 4.11. If R is Gorenstein in characteristic $p > 0$, then BCM-regularity for all BCM algebras is equivalent to strong F-regularity. .

Question 4.12 (Characteristic zero comparison)**.** Let R be Gorenstein of characteristic zero. Is it true that

$$R \text{ is BCM-regular for all BCM algebras} \quad \iff \quad R \text{ has rational singularities?}$$

One implication is known:

$$\text{BCM-regular for all } B \quad \Longrightarrow \quad \text{rational singularities.}$$

Examples and Sources of BCM Algebras

Characteristic $p > 0$

In characteristic $p > 0$, one important source of BCM algebras is the absolute integral closure R^+. The perfect closure R_{perf} is almost BCM.

Mixed Characteristic

In mixed characteristic, the following completions of the absolute integral closure are BCM:

$$(R^+)^{\wedge p}, \qquad (R^+)^{\wedge \mathfrak{m}}.$$

Characteristic Zero

In characteristic zero, various constructions of BCM algebras are available. The lecture notes do not specify a single preferred construction here.

Equational Lemmas

Lemma 4.13 (Equational lemma; Huneke–Lyubeznik)**.** *Let R be a domain of characteristic $p > 0$, and let $J \subseteq R$ be an ideal. Let*

$$\eta \in H^i_J(R).$$

Suppose that the set

$$\{\eta, \eta^p, \eta^{p^2}, \dots\}$$

generates a finitely generated R-module. Then there exists a finite domain extension

$$R \hookrightarrow S$$

such that the image of η vanishes:

$$\eta \longmapsto 0 \quad \textit{in} \quad H^i_{JS}(S).$$

Remark 4.14. If

$$\eta \in H^i_{\mathfrak{m}}(R^+) \qquad \text{for } i < \dim R,$$

then η is already represented over some finite extension R' of R. If the equational lemma applies, then after passing to a further finite extension one kills η. This is the mechanism behind the vanishing

$$H^i_{\mathfrak{m}}(R^+) = 0 \qquad \text{for } i < \dim R$$

in characteristic $p > 0$.

Lemma 4.15 (Equational lemma II; Hochster–Huneke–Bhatt)**.** *Let $Y \to \operatorname{Spec} R$ be* *proper, where R is a Noetherian integral domain of characteristic $p > 0$. For every class*

$$\eta \in H^i(Y, \mathcal{O}_Y), \qquad i > 0,$$

there exists a finite surjective integral morphism

$$Y' \to Y$$

such that the image of η in

$$H^i(Y', \mathcal{O}_{Y'})$$

is zero.

Bhatt's Theorem and Derived Splinters

Theorem 4.16 (Bhatt)**.** *Let R be an excellent domain of characteristic $p > 0$, and let*

$$Y \longrightarrow \operatorname{Spec} R$$

be proper and surjective. There is a factorization in the derived category

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend right, "1\mapsto 1"'] & R\Gamma(Y, \mathcal{O}_Y) \arrow[r] & R^+. \end{tikzcd}$$

In particular, after passing to suitable finite covers, one can kill the higher cohomology classes appearing in $R\Gamma(Y, \mathcal{O}_Y)$.

Definition 4.17 (Splinters and derived splinters)**.** Let R be a domain.

(1) R is a **splinter** if every finite extension

$$R \hookrightarrow S$$

splits as a map of R-modules. Equivalently, $R \to R^+$ is pure.

(2) R is a **derived splinter** if, for every proper surjective morphism

$$Y \longrightarrow \operatorname{Spec} R,$$

the natural map

$$R \longrightarrow R\Gamma(Y, \mathcal{O}_Y)$$

splits in the derived category.

(3) R is a **birational derived splinter** if the same splitting condition holds for every blowup

$$Y \longrightarrow \operatorname{Spec} R.$$

Remark 4.18. In characteristic $p > 0$, Bhatt's theorem implies

$$\text{splinter} \quad \Longrightarrow \quad \text{derived splinter}.$$

Remark 4.19. Bhargav Bhatt gave a mixed characteristic version of the equational lemma and deduced that

$$(R^+)^{\wedge p}$$

is BCM. In mixed characteristic, one also has a factorization

$$R \longrightarrow R\Gamma(Y, \mathcal{O}_Y) \longrightarrow (R^+)^{\wedge p}.$$

Implication Diagrams

Characteristic $p > 0$

In characteristic $p > 0$, the relationships discussed in the lecture are summarized by the following diagram: All rings are assumed Noetherian, excellent, local domains where appropriate. The arrow splinter ⇒ F-split assumes F-finiteness. The dashed arrow to strong F-regularity is known under additional hypotheses such as Gorenstein or $\mathbb{Q}$-Gorenstein.

$$\begin{tikzcd}
\text{strongly } F\text{-regular} \arrow[r, Rightarrow] & \text{BCM-regular for all } B \arrow[r, Rightarrow] & \text{splinter} \arrow[r, Rightarrow, "F\text{-finite}"] \arrow[dl, Rightarrow, dashed, "?"', "\text{known if Gorenstein}"] \arrow[d, equal] \arrow[dr, Rightarrow] & F\text{-split} \\
& \text{strongly } F\text{-regular} & \text{derived splinter} & \text{normal and CM.}
\end{tikzcd}$$

Remark 4.20 (Open question)**.** Does

$$\text{splinter} \quad \Longrightarrow \quad \text{strongly } F\text{-regular}$$

hold in characteristic $p > 0$? This is known when R is Gorenstein.

The Cohen–Macaulayness mechanism is reflected in the vanishing

$$\boxed{H^i_{\mathfrak{m}}(R) \hookrightarrow H^i_{\mathfrak{m}}(R^+) = 0 \qquad \text{for all } i < \dim R.}$$

Remark 4.21. The injection $H^i_{\mathfrak{m}}(R) \hookrightarrow H^i_{\mathfrak{m}}(R^+)$ comes from purity of $R \to R^+$ when R is a splinter. The vanishing of the right-hand side is the big Cohen–Macaulay property of R^+ in characteristic $p > 0$. Thus splinters in positive characteristic are Cohen–Macaulay.

Characteristic Zero

In characteristic zero, the lecture records the following relationships:

$$\begin{tikzcd}
\begin{array}{c}\text{BCM-regular}\\ \text{for all } B\end{array} \arrow[r, Rightarrow] & \text{rational} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \begin{array}{c}\text{derived}\\ \text{splinter}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{birational}\\ \text{derived}\\ \text{splinter}\end{array} \\
 & \text{normal} & \text{splinter} &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{log-terminal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{rational} \\
\text{log-canonical} &
\end{tikzcd}$$

Conjecture 4.22. In characteristic zero,

$$\text{rational singularities} \quad \Longrightarrow \quad \text{BCM-regular for all BCM algebras.}$$

This is known if R is Gorenstein.

Remark 4.23. The equality between rational singularities and derived splinters in characteristic zero is usually understood through resolution of singularities and Grauert–Riemenschneider-type vanishing. The birational derived splinter condition tests the same splitting only on birational proper maps, such as blowups.

Mixed Characteristic

In mixed characteristic, we have:

$$\text{BCM-regular for all } B \quad \Longrightarrow \quad \text{splinter} \quad \Longleftrightarrow \quad \text{derived splinter.}$$

The corresponding diagram is

$$\begin{tikzcd}
\text{BCM-regular for all } B \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{splinter} \arrow[r, Rightarrow] & \text{normal and Cohen–Macaulay} \\
R \to (R^+)^{\wedge p} \text{ is pure} & &
\end{tikzcd}$$

Remark 4.24. The mixed characteristic theory is subtler because Frobenius methods are not directly available. Perfectoid methods and the construction of $(R^+)^{\wedge p}$ as a BCM algebra supply the replacement for the positive characteristic argument.

5 Lecture-5 May 22, 2026

Let R be an integral domain, and let $J \subseteq R$ be a finitely generated ideal. We denote the integral closure of J by $\overline{J}$.

Definition 5.1 (Integral closure of an ideal)**.** An element $x \in R$ lies in $\overline{J}$ if and only if x satisfies an equation of integral dependence over J, namely

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in J^i.$$

Equivalently, let

$$Y \longrightarrow \operatorname{Spec} R$$

be the normalized blowup of J. Then $J \cdot \mathcal{O}_Y$ is the ideal sheaf obtained by expanding J to each affine chart of Y. In this language,

$$\overline{J} = \Gamma(Y, J \cdot \mathcal{O}_Y) \cap R.$$

This describes the integral closure of J after passing to the normalization.

Remark 5.2. If $Y \to \operatorname{Spec} R$ is the normalized blowup of J, then $J\mathcal{O}_Y$ becomes an invertible ideal sheaf on Y. The sections of this sheaf encode precisely the elements whose valuations along the exceptional divisors are at least the valuations of J. This is the valuative interpretation of integral closure.

Briançon–Skoda Type Theorems

Theorem 5.3 (Briançon–Skoda)**.** *Let*

$$I \subseteq \mathbb{C}[x_1, \dots, x_d]$$

be an n-generated ideal. Then

$$\overline{I^{n+k-1}} \subseteq I^k \subseteq \overline{I^k} \qquad \text{for all } k \geq 0,\ k \in \mathbb{Z}.$$

Remark 5.4. The Briançon–Skoda theorem gives uniform control over integral closures of powers of an ideal in terms of ordinary powers. The containment is nontrivial for $k \geq 1$; for $k = 0$, the displayed statement is formally harmless because $I^0 = R$.

Theorem 5.5 (Lipman–Sathaye)**.** *Let R be a regular ring. If $I \subseteq R$ is n-generated, then*

$$\overline{I^{n+k-1}} \subseteq I^k$$

for all integers $k \geq 1$.

Theorem 5.6 (Hochster–Huneke)**.** *Let R be a Noetherian excellent domain of characteristic $p > 0$, and let $I \subseteq R$ be an n-generated ideal. Then*

$$\overline{I^{n+k-1}} \subseteq I^k R^+ \cap R = (I^k)^+,$$

where R^+ denotes the absolute integral closure of R, and $(I^k)^+$ denotes the plus closure of I^k. If R is a splinter, then

$$(I^k)^+ = I^k.$$

Remark 5.7. Recall that R^+ is the integral closure of R inside an algebraic closure of its fraction field. The plus closure of an ideal $Q \subseteq R$ is

$$Q^+ = QR^+ \cap R.$$

For splinters, finite extensions split as R-modules; this splitting forces plus closure to coincide with the original ideal in many excellent characteristic-p settings.

Theorem 5.8 (Heitmann, Schwede–Rodriguez-Villalobos)**.** *In mixed characteristic, one* *has a containment of the form*

$$\overline{I^{n+k-1}} \subseteq I^k (R^+)^{\wedge p} \cap R,$$

where $(R^+)^{\wedge p}$ *denotes the* p*-adic completion of* R^+*.*

Derived Expansions and Contractions

Instead of expanding and contracting ideals only along ordinary ring maps, one can expand and contract using derived global sections. The guiding object is

$$\mathbf{R}\Gamma(Y, \mathcal{O}_Y),$$

where $Y \to \operatorname{Spec} R$ is a blowup, normalized blowup, or resolution. For an ordinary R-algebra B, one has

$$I \subseteq R, \qquad IB \cap R = \ker(R \to B/IB).$$

Option 1: Koszul Complex Closures

Let

$$I = (f_1, \ldots, f_n).$$

Let

$$\operatorname{Kos}(f) = \operatorname{Kos}(f_1, \ldots, f_n)$$

be the Koszul complex on the chosen generators. Consider the natural map

$$R \longrightarrow \operatorname{Kos}(f) \otimes_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y).$$

One defines a closure operation by taking the kernel of the induced map on degree-zero homology:

$$\ker\left(R \longrightarrow H_0\Big(\operatorname{Kos}(f) \otimes_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big)\right).$$

Equivalently, this kernel may be viewed as an annihilator of the relevant degree-zero Koszul homology:

$$\operatorname{Ann}_R\left(H_0\Big(\operatorname{Kos}(f) \otimes_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big)\right).$$

This construction is independent of the chosen generators of I. This is associated with work of Epstein–McDonald–R.G.–Schwede.

Definition 5.9 (Koszul-homological closure)**.** For a morphism $Y \to \operatorname{Spec} R$, define

$$I^{\mathrm{KH}_Y} := \ker\left(R \longrightarrow H_0\Big(\operatorname{Kos}(f) \otimes_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\Big)\right).$$

The notation records that the closure is defined using Koszul homology and the space Y.
.

Theorem 5.10 (Ma–McDonald–R.G.–Schwede)**.** *Let $I \subseteq R$ be an n-generated ideal. Then*

$$\overline{I^{n-1}} \subseteq I^{\mathrm{KH}_Y},$$

where Y is the blowup of I^{n-1}, the normalized blowup, or a resolution of singularities.

Remark 5.11. For $k > 1$, the analogous statement is false; it is easy to find examples.

Option 2: Derived Closure Using R/I

Define

$$I^Y := \ker\Big(R \longrightarrow H_0\big(R/I \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)\big)\Big).$$

This closure operation is hard to work with directly.

Question 5.12. Is the closure I^Y idempotent? Equivalently, is

$$(I^Y)^Y = I^Y$$

for Y a blowup, resolution, or related birational model?

Option 3: Buchsbaum–Eisenbud and Eagon–Northcott Complexes

Let

$$I = (f_1, \dots, f_n).$$

Let

$$\mathrm{BE}_k(f)$$

denote the relevant Buchsbaum–Eisenbud complex, equivalently the appropriate Eagon–Northcott complex. For $k = 1$, this is the Koszul complex:

$$\mathrm{BE}_1(f) = \mathrm{Kos}(f).$$

If $(f_1, \dots, f_n)$ is a regular sequence, then $\mathrm{BE}_k(f)$ resolves R/I^k. Consider the map

$$\mathbb{Z}[x_1, \dots, x_n] \longrightarrow R, \qquad x_i \longmapsto f_i.$$

Let $C_\bullet$ be a minimal free resolution of

$$\frac{\mathbb{Z}[x_1,\dots,x_n]}{(x_1,\dots,x_n)^k}.$$

After tensoring with R, one obtains a complex quasi-isomorphic to the Buchsbaum–Eisenbud complex:

$$C_\bullet \otimes_{\mathbb{Z}[x_1,\dots,x_n]} R \simeq \mathrm{BE}_k(f).$$

Definition 5.13 (Buchsbaum–Eisenbud derived closure)**.** Define

$$(I^k)^{\mathrm{BE}_k(f),Y} := \ker\Big(R \longrightarrow H_0\big(\mathrm{BE}_k(f) \otimes_R \mathbf{R}\Gamma(Y,\mathcal{O}_Y)\big)\Big).$$

Theorem 5.14 (MMRGS)**.** *Let R be any ring, and let $I \subseteq R$ be an n-generated ideal. Then*

$$\overline{I^{n+k-1}} \subseteq (I^k)^{\mathrm{BE}_k(f),Y},$$

where Y is the blowup of I^{n+k-1}, the normalized blowup of I, or a resolution.

Remark 5.15. The three options above define homological closure operations by replacing the ordinary quotient R/I^k with complexes that compute it in favorable cases. This derived viewpoint produces Briançon–Skoda type containments even when ordinary flatness or regular-sequence hypotheses fail.

Since the natural map

$$R \longrightarrow \mathrm{BE}_k(f) \longrightarrow R/I^k$$

factors through R/I^k, one has the containment

$$(I^k)^{\mathrm{BE}_k(f),Y} \subseteq (I^k)^Y.$$

Derived Birational Splinters

Corollary 5.16. *Suppose R is a birational derived splinter. That is, for every relevant blowup $Y \to \operatorname{Spec} R$, the natural map*

$$R \xrightarrow{\mathrm{id}} \mathbf{R}\Gamma(Y,\mathcal{O}_Y)$$

admits a retraction in the derived category:

$$\mathbf{R}\Gamma(Y,\mathcal{O}_Y) \longrightarrow R.$$

Then, for every n-generated ideal $I \subseteq R$,

$$\overline{I^{n+k-1}} \subseteq I^k.$$

Proof. Tensor the splitting

$$R \longrightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \longrightarrow R$$

with the Buchsbaum–Eisenbud complex $\mathrm{BE}_k(f)$. This gives a splitting after applying

$$\mathrm{BE}_k(f) \otimes_R -.$$

Consequently,

$$\ker\left(R \to H_0(\mathrm{BE}_k(f) \otimes_R R)\right) \cong (I^k)^{\mathrm{BE}_k(f),Y}.$$

Since $\mathrm{BE}_k(f)$ computes R/I^k in the relevant derived sense, the kernel on the left is I^k. Combining this with the MMRG containment gives

$$\overline{I^{n+k-1}} \subseteq (I^k)^{\mathrm{BE}_k(f),Y} = I^k.$$

□

What is a derived birational splinter?

In characteristic $p > 0$, or in mixed characteristic, splinter-type conditions are encoded by diagrams of the form

$$\begin{tikzcd} R \arrow[r, dashed] & \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \arrow[r] & R^+. \end{tikzcd}$$

Here R^+ is the absolute integral closure. The dashed arrow indicates a derived splitting or factorization associated to the birational model Y. There are implications

$$F\text{-rational} \Longrightarrow \text{pseudo-rational} \Longrightarrow \text{derived birational splinter}.$$

The converses do not hold in general:

$$F\text{-rational} \not\Longleftarrow \text{pseudo-rational} \not\Longleftarrow \text{derived birational splinter}.$$

Remark 5.17. A derived birational splinter is a derived analogue of a splinter, but only tested against proper birational maps rather than arbitrary finite maps. The condition asks that R splits off from $\mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ in the derived category.

Proof in the Case $k = 1$

Let Y be the normalized blowup of I, where

$$I = (f_1, \ldots, f_n).$$

Consider the Koszul complex

$$\mathrm{Kos}(f, \mathcal{O}_Y).$$

Let

$$\mathcal{L} := (f_1, \ldots, f_n)\mathcal{O}_Y = I\mathcal{O}_Y = \mathcal{O}_Y(-E).$$

Thus $\mathcal{L}$ is the invertible sheaf corresponding to the exceptional divisor E. The proof uses three complexes of sheaves, denoted $A_\bullet$, $B_\bullet$, and $C_\bullet$.

The three complexes

The first complex is the Koszul complex on $(f_1, \ldots, f_n)$ after pulling back to Y:

$$0 \longrightarrow \bigoplus_{\binom{n}{n}} \mathcal{O}_Y \longrightarrow \bigoplus_{\binom{n}{n-1}} \mathcal{O}_Y \longrightarrow \cdots \longrightarrow \bigoplus_{\binom{n}{1}} \mathcal{O}_Y \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_Y \longrightarrow 0.$$

Call this complex $A_\bullet$. The second complex is built out of powers of the invertible sheaf $\mathcal{L}$:

$$0 \longrightarrow \mathcal{L}^n \longrightarrow \bigoplus \mathcal{L}^{n-1} \longrightarrow \cdots \longrightarrow \mathcal{L} \longrightarrow 0.$$

Call this complex $B_\bullet$. It is exact everywhere. The third complex $C_\bullet$ records the leftmost term:

$$0 \longrightarrow \mathcal{L}^n \longrightarrow 0.$$

Since

$$\mathcal{L}^n = I^n \mathcal{O}_Y,$$

we may also write the nonzero term of $C_\bullet$ as $I^n\mathcal{O}_Y$. The inclusions of powers of $\mathcal{L}$ into the corresponding trivial summands of the Koszul complex give a diagram of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \underset{\binom{n}{n}}{\bigoplus} \mathcal{O}_Y \arrow[r] & \underset{\binom{n}{n-1}}{\bigoplus} \mathcal{O}_Y \arrow[r] & \cdots \arrow[r] & \underset{\binom{n}{1}}{\bigoplus} \mathcal{O}_Y \arrow[r, "(f_1,\ldots,f_n)"] & \mathcal{O}_Y \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_Y \arrow[r] \arrow[u, equal] & \underset{\binom{n}{n-1}}{\bigoplus} \mathcal{L} \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & \underset{\binom{n}{1}}{\bigoplus} \mathcal{L}^{n-1} \arrow[r] \arrow[u, hook] & \mathcal{L}^n \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{L}^n \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
 & & & & & I^n\mathcal{O}_Y &
\end{tikzcd}$$

In the lecture notation, the top row is $A_\bullet$, the middle row is $B_\bullet$, and the bottom row is $C_\bullet$.

Applying derived global sections

Apply

$$\mathbf{R}\Gamma(Y, -)$$

to the morphisms of complexes

$$C_\bullet \longrightarrow B_\bullet \longrightarrow A_\bullet.$$

This gives

$$\mathbf{R}\Gamma(Y, I^n\mathcal{O}_Y) \longrightarrow \mathbf{R}\Gamma(Y, B_\bullet) \longrightarrow \mathrm{Kos}(f) \otimes_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y).$$

The term

$$\mathbf{R}\Gamma(Y, I^n\mathcal{O}_Y)$$

acts as the derived integral closure of I^n. Since $B_\bullet$ is exact, one has

$$\mathbf{R}\Gamma(Y, B_\bullet) \simeq 0.$$

Taking H_0, the map from

$$\Gamma(Y, I^n\mathcal{O}_Y)$$

to the degree-zero homology of the Koszul-derived complex becomes zero. Since

$$\overline{I^n} \subseteq \Gamma(Y, I^n\mathcal{O}_Y) \cap R,$$

we obtain the desired containment in the Koszul-homological closure:

$$\overline{I^n} \subseteq I^{\mathrm{KH}_Y}.$$

Remark 5.18. The key geometric point is that on the normalized blowup, $I\mathcal{O}_Y$ becomes invertible. This allows the Koszul complex to be compared with a complex built from powers of the line bundle $\mathcal{L} = I\mathcal{O}_Y$, and the exactness of the latter forces elements coming from $\overline{I^n}$ to vanish in the relevant degree-zero Koszul homology.

Uniform Briançon–Skoda Type Consequence

Corollary 5.19. *Let R be quasi-excellent, reduced, and finite dimensional. Then there exists an element $c > 0$ such that*

$$\overline{I^{k+c}} \subseteq I^k \qquad \textit{for all ideals } I \subseteq R.$$

Diagram of implication. The relevant obstruction is measured by a map

$$\begin{tikzcd}
R \arrow[r] \arrow[dr, "g"'] & \mathbf{R}\Gamma(Y, \mathcal{O}_Y) \arrow[d, "\text{some map}"', "\neq 0"] \\
 & R.
\end{tikzcd}$$

This gives an element g such that

$$g \cdot \overline{I^{n+k-1}} \subseteq I^k.$$

Uniformly controlling such maps yields a constant $c > 0$ with

$$\overline{I^{k+c}} \subseteq I^k$$

for all ideals $I \subseteq R$. □

Remark 5.20. The statement is a uniform Briançon–Skoda type theorem. The constant c depends on the singularities of R, not on the particular ideal I. In regular rings one can take $c = n - 1$ for an n-generated ideal; in singular rings, the correction term records the failure of the relevant derived splitting.

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