

PASCA 2.0

Summer School at CIMAT



CIMAT

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Lecture Notes

Vertex decomposability and F -singularities

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1 Geometric Vertex Decomposition and F-singularities

Introduction and Setting

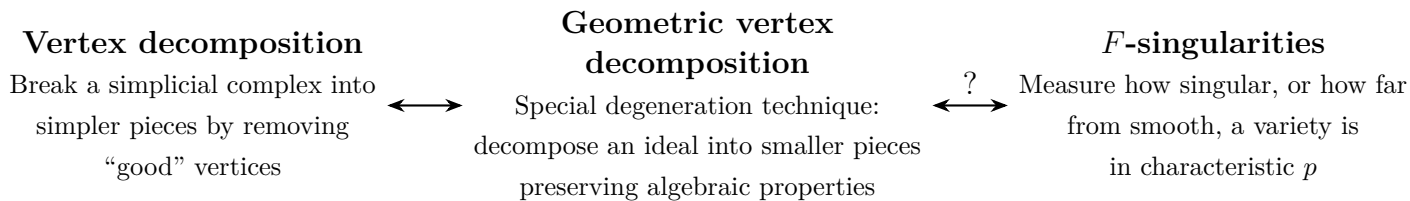
Throughout this presentation, we fix the following algebraic setting:

- $S = \mathbb{K}[x_1, \dots, x_n]$ is a standard graded polynomial ring.
- \mathbb{K} is a perfect field.
- $I \subseteq S$ is a homogeneous ideal.

The main topics covered in this talk are:

1. Geometric vertex decomposable (GVD) ideals.
2. Frobenius singularities, also called F -singularities.
3. How singularities behave when we have a geometric vertex decomposition.

Conceptual overview. The core idea is to find a geometric analogue of the combinatorial concept of vertex decomposition. In combinatorics, one breaks a simplicial complex into simpler pieces by removing “good” vertices. Geometrically, one wants a special degeneration technique that decomposes an ideal into smaller pieces while preserving algebraic properties. Ultimately, we want to understand how this technique interacts with F -singularities, which measure how singular, or how far from smooth, a variety is in characteristic p .

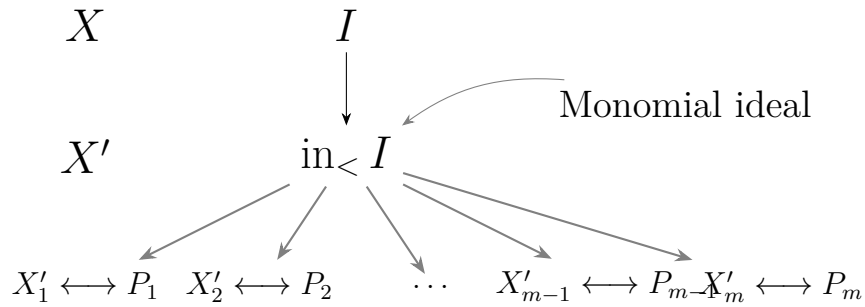


When a geometric vertex decomposition is applied, an object is split into two simpler pieces. A major question is whether properties such as F -split, F -rational, or F -injective pass to these pieces and conversely.

Question: How do singularities behave when we have a GVD?

Geometric Vertex Decomposition: Motivation

Motivation. One common way to degenerate an ideal is to take the initial ideal with respect to a chosen term order. However, this classical approach often fragments the ideal too drastically.



Remark 1.1. This standard degeneration creates many pieces all at once. Furthermore, the degenerated space X' might be non-reduced, which complicates geometric analysis. This motivates the search for a more controlled degeneration.

I. Geometric Vertex Decomposition [Knutson–Miller–Yong]

To remedy the issues with general term-order degeneration, Knutson, Miller, and Yong introduced geometric vertex decomposition. This technique degenerates the ideal with respect to a single coordinate, splitting it into exactly two pieces.

Setting. Let

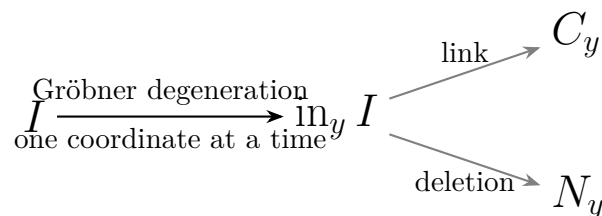
$$S := \mathbb{K}[x_1, \dots, x_n], \quad y = x_i,$$

and let $I \subseteq S$. For the purposes of this lecture, assume that I is homogeneous and unmixed.

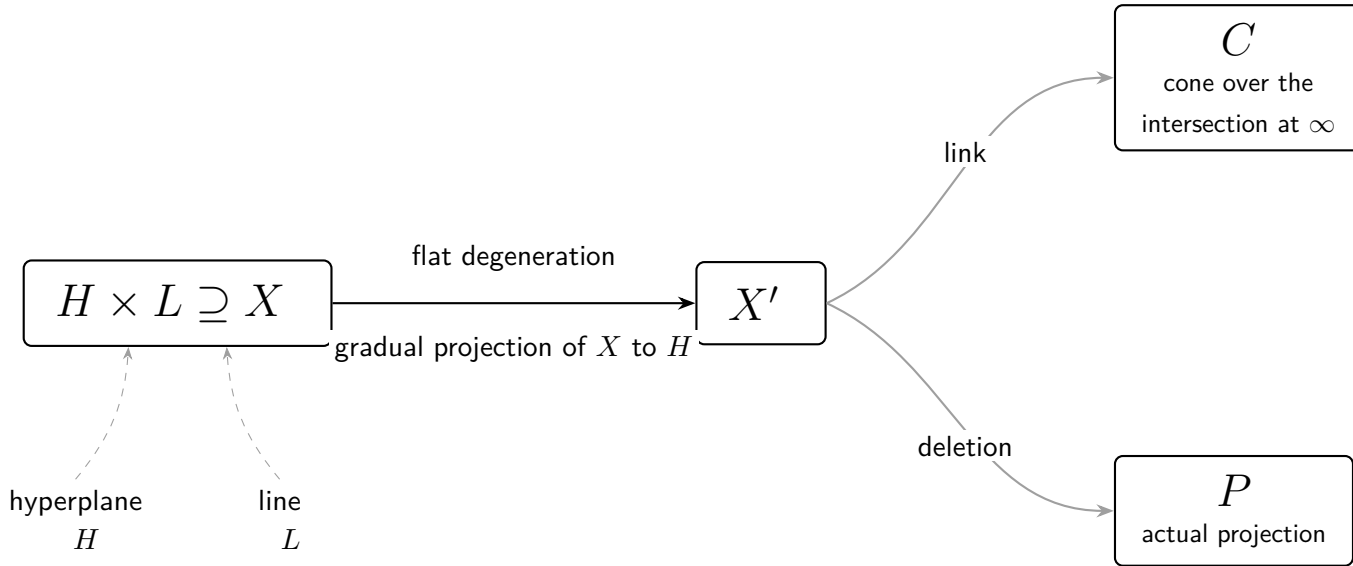
Theorem 1.2 (Knutson–Miller–Yong). *Assume that I is squarefree in y . Then one can degenerate I to I' and decompose it into two smaller, simpler ideals, called the link and the deletion. The invariants of I can be derived from the invariants of these two pieces. This decomposition is called a geometric vertex decomposition at y .*

This controlled decomposition can be viewed from two perspectives: algebraic and geometric.

Algebraic point of view.



Geometric point of view.



Formal Definition of GVD

We now make the construction rigorous.

Setting. Let

$$S := \mathbb{K}[x_1, \dots, x_n], \quad y = x_n,$$

and use the lexicographic order $<_{\text{lex}}$.

Assume that I is homogeneous and unmixed and has a Gröbner basis of the form

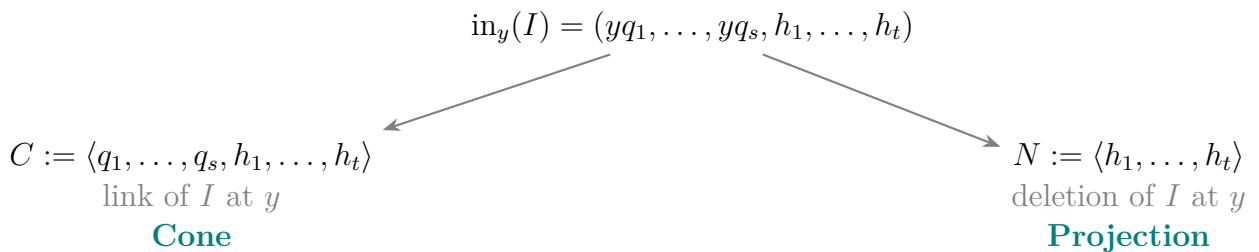
$$\mathcal{G}_I = \{yq_1 + r_1, \dots, yq_s + r_s, h_1, \dots, h_t\},$$

where y does not divide any term of any q_i , r_i , or h_j . Equivalently, no y occurs where it is not explicitly displayed.

Taking the initial ideal with respect to y , one obtains

$$X' \longleftrightarrow \text{in}_y(I) = (yq_1, \dots, yq_s, h_1, \dots, h_t).$$

This degenerated ideal breaks into two simpler components:



Algebraically, the initial ideal can be reconstructed from the link and the deletion as follows:

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle).$$

Geometrically, this corresponds to the union of the cone and the projection:

$$X' = C \cup P.$$

This exact decomposition is what we call a **geometric vertex decomposition**.

Recursive Definition

Definition 1.3 (KR, 2021). An unmixed ideal I is *geometrically vertex decomposable* if either

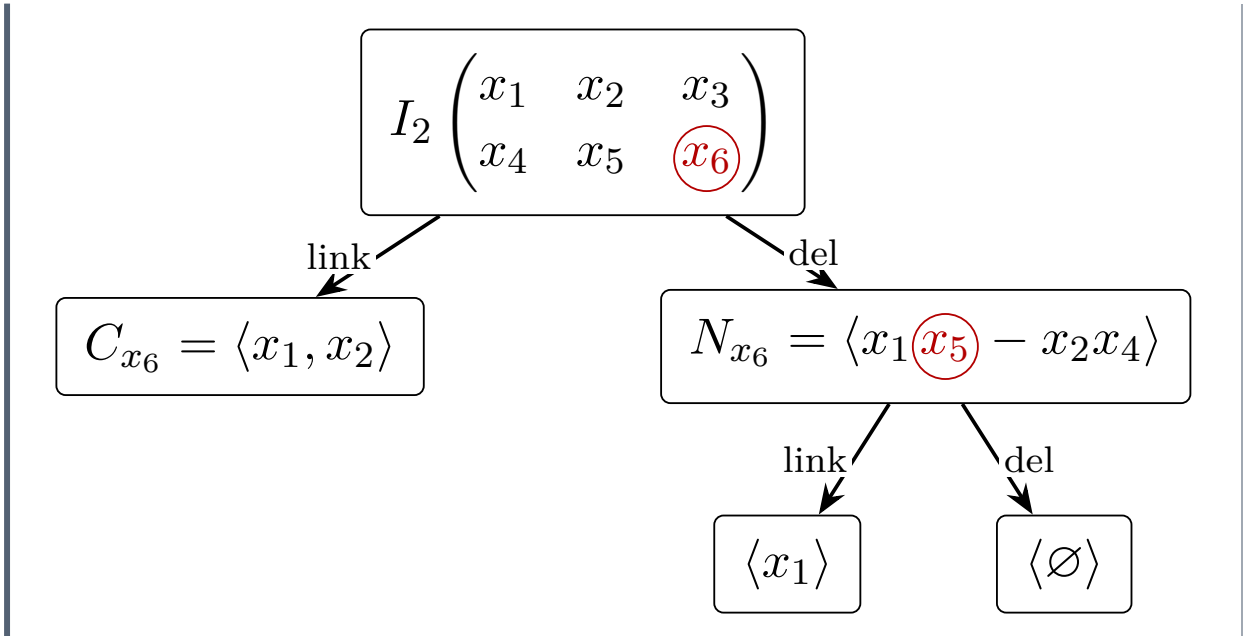
- $I = \langle 1 \rangle$ or $I = \langle \text{variables} \rangle$; or
- there exists a variable y such that

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

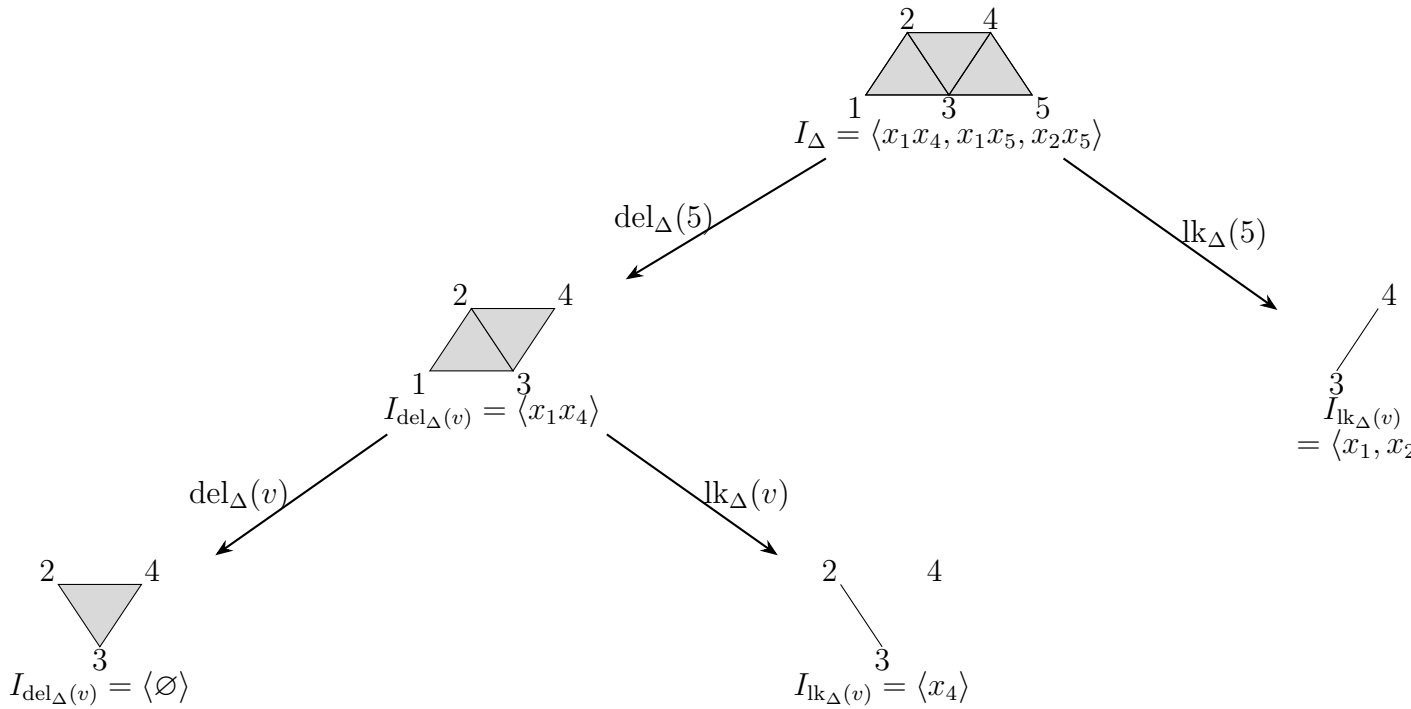
and $C_{y,I}$ and $N_{y,I}$ are unmixed and geometrically vertex decomposable in the smaller ring.

Example 1.4 (Two-minors).

$$I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix} = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5).$$



Remark 1.5. The following diagram illustrates the geometric vertex decomposition of a Stanley–Reisner ideal corresponding to a simplicial complex Δ . Deleting a variable corresponds to taking the deletion subcomplex, and linking corresponds to the link subcomplex. The vertex v in the subscripts is a placeholder for the vertex being processed at the given branching step.



For squarefree monomial ideals:
 GVD = Δ vertex decomposable.

Some results, mostly in [KMY] and [KR, 2021].

- Matrix Schubert varieties, Stanley–Reisner ideals of vertex decomposable complexes Δ , toric ideals of certain graphs, and determinantal ideals are GVD.
- GVD ideals are radical. If I is radical, then C and N are radical.
- GVD ideals are Cohen–Macaulay.
- $\text{ht}(C) = \text{ht}(I) = \text{ht}(N) + 1$.
- GVD can be used to prove that a conjectured Gröbner basis is in fact a Gröbner basis.

II. F-singularities

Let

$$R = S/I$$

be a Noetherian ring of characteristic $p > 0$.

Frobenius map:

$$\begin{aligned} F: R &\longrightarrow F_*R, \\ r &\longmapsto r^p. \end{aligned}$$

Algebraic properties of F

\iff

Singularities of the variety

F is injective

\iff

R is reduced

F -split (\leftrightarrow log-canonical)

F -rational (\leftrightarrow rational)

F is flat

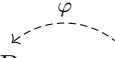
$\xleftrightarrow{\text{Kunz, 1969}}$

R is regular, equivalently smooth

II. F-singularities: F-split

Let $R(= S/I)$ be a Noetherian ring of characteristic $p > 0$.

$$F : R \longrightarrow F_*R$$




Frobenius map:

$$r \longmapsto r^p$$

Definition 1.6 (Mehta-Ramanathan). R is F -split if $\exists \varphi : F_*R \rightarrow R$ such that $\varphi \circ F = 1_R$. Such φ is called an F -splitting.

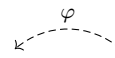
Another interpretation:

- R regular $\iff F$ is flat $\iff F_*R$ free as R -module
- R F -split $\iff F$ splits $\iff F_*R = R \oplus \dots$


 a copy of R as direct summand

Explanation: In characteristic $p > 0$, the Frobenius endomorphism $F(r) = r^p$ is a central tool. By Kunz's Theorem, a local ring is regular if and only if the Frobenius map is flat. The notion of F -splitting is a weakening of this property. If the map $R \rightarrow F_*R$ splits, then R is an F -split ring. Geometrically, F -split varieties have mild singularities and enjoy strong vanishing theorems.

$$F : R \longrightarrow F_*R$$


 How do we find φ ??

Example 1.7. $S = \mathbb{K}[x_1, \dots, x_n]$ is F -split. The F -splittings of S are of the form $\varphi_f := \text{Tr}(f \bullet)$. When $f = (x_1 \cdots x_n)^{p-1}$, the corresponding map $\text{Tr}((x_1 \cdots x_n)^{p-1} \bullet)$ is called *standard splitting*.

We want an F -splitting for $R = S/I$

Definition 1.8. If I is an ideal such that $\varphi_f(I) \subseteq I$, then φ_f induces a splitting on S/I and we say that I is *compatibly split* by φ_f .

Explanation: The operator Tr generally denotes the trace map, which serves as a generator for the module of homomorphisms $\text{Hom}_S(F_*S, S)$. To pass an F -splitting from the ambient polynomial ring S to a quotient S/I , the ideal I must be stable under the splitting map. A compatible splitting heavily restricts the geometry of the variety defined by I .

We focus on splittings of the form $\varphi_f := \text{Tr}(f^{p-1}\bullet)$ with $\text{in}_<(f)$ squarefree.

We can obtain many compatibly split ideals:

(f) is c.s. \rightarrow irreducible components are c.s. $\rightarrow +, \cap, \dots$ etc

Remark 1.9. All these are compatibly split by φ_f and they enjoy very nice properties!

How do we choose f ?

For some ideals there is a "natural" choice...

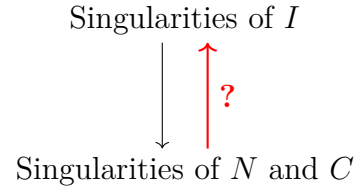
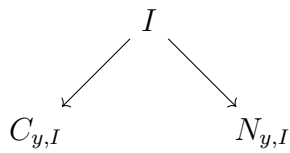
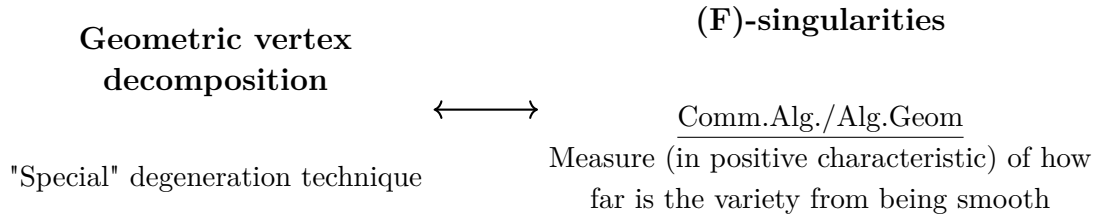
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \quad H = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \\ x_3 & x_4 & x_5 & x_6 \\ x_4 & x_5 & x_6 & x_7 \end{pmatrix}$$

$$f = x_{31} \det M_1 \det X \det M_2 x_{13}$$

$$f = \det H \det M_1$$

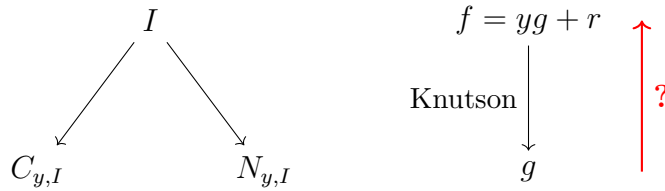
Remark 1.10 (IDEA). If I has "enough" structure, geometric vertex decomposition provides a method to break down the ideals into "smaller" ideals from which we can recursively build an F -splitting of I .

Explanation: Here X represents a generic matrix and H a Hankel matrix. Determinantal ideals associated with these matrices often have compatible splittings defined by taking f to be the product of certain principal minors. Geometric Vertex Decomposition (GVD) generalizes classical vertex decomposability from simplicial complexes to algebraic structures, providing an inductive framework to build splittings from smaller, simpler components.



III. GVD & F-splittings

Theorem 1.11 (Knutson '09). *If φ_f compatibly splits $I \implies \varphi_{\text{in}_y(f)}$ compatibly splits $C_{y,I}, N_{y,I}$.*



Theorem 1.12 (DNG+'25). *Assume that I has a GVD at y and C_y, N_y are compatibly split by φ_g . If $\exists u \mid g$ such that $u \in C_y$ and NZD on N_y , then $\exists f = yg + r$ such that I is compatibly split by φ_f .*

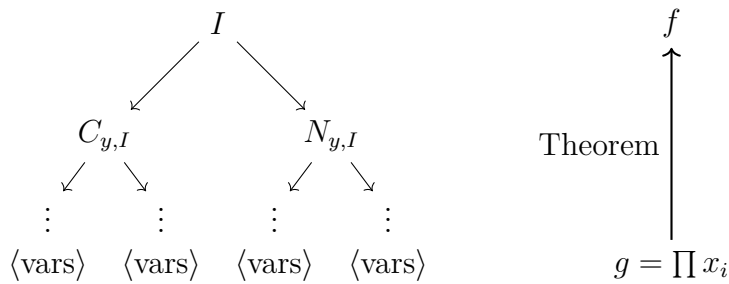
Explanation: The ideal $C_{y,I}$ acts as the "vertex" component and $N_{y,I}$ as the "link" component of the degeneration of I . Knutson's result establishes the downward implication: a splitting on I descends to its GVD components. The theorem by DNG+'25 successfully reverses this arrow (answering the question mark), giving precise conditions under which a compatible splitting of the components C_y and N_y can be lifted back to I , explicitly constructing the polynomial f that realizes the F -splitting.

III. GVD & F-splittings

Theorem 1.13 (DNG+, '25). *Assume that I has a GVD at y and C_y, N_y are compatibly split by φ_g . If $\exists u \mid g$ such that $u \in C_y$ and NZD on N_y , then $\exists f = yg + r$ such that I is compatibly split by φ_f .*

Constructive proof: f can be obtained as $f = \frac{v}{u}g$ with $\text{in}_{<}v = yu$

How to use the theorem: Assume that I is GVD.



Examples: determinant ideals, (some) toric ideals of graph, graded lower bound cluster algebras...

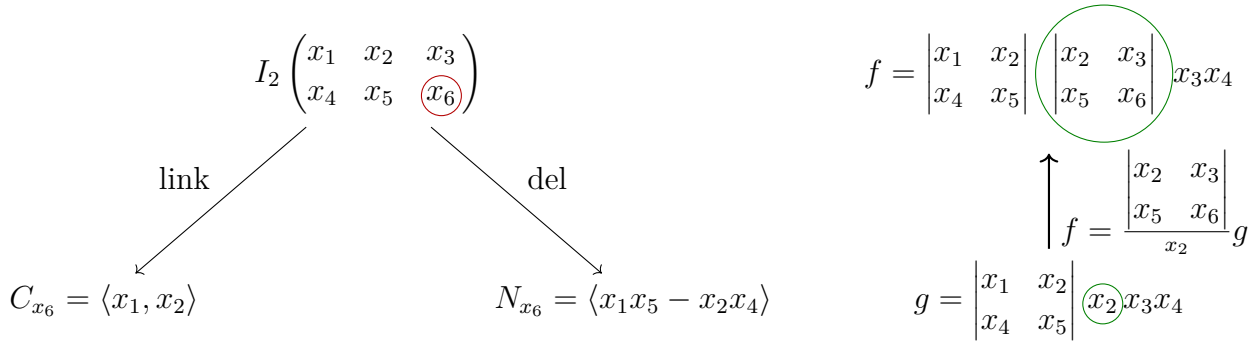
Remark 1.14. The diagram illustrates the recursive application of the theorem. Ideals iteratively decompose via links ($C_{y,I}$) and deletions ($N_{y,I}$) down to monomial ideals generated by variables, while the splitting polynomial is constructed bottom-up starting from the product of the variables $g = \prod x_i$.

III. GVD & F-splittings (Continued)

Theorem 1.15 (DNG+, '25). *Assume that I has a GVD at y and C_y, N_y are compatibly split by φ_g . If $\exists u \mid g$ such that $u \in C_y$ and NZD on N_y , then $\exists f = yg + r$ such that I is compatibly split by φ_f .*

Constructive proof: f can be obtained as $f = \frac{v}{u}g$ with $\text{in}_{<}v = yu$

Example (2-minors):



Remark 1.16. In this determinantal ideal example, $y = x_6$. The minor $v = \begin{vmatrix} x_2 & x_3 \\ x_5 & x_6 \end{vmatrix} = x_2x_6 - x_3x_5$ has initial term $\text{in}_{<v} = x_2x_6$, meaning $u = x_2$. The theorem guarantees we can replace the circled x_2 in g with the circled minor in f .

Two obstructions:

1. Sometimes it is not possible to find a common splitting for C_y and $N_{y,I}$
(If I is not F -split, the procedure tells us at which level this fails).
2. The condition “ $\exists u \mid g$ such that $u \in C_y$ and NZD on N_y ” is necessary!

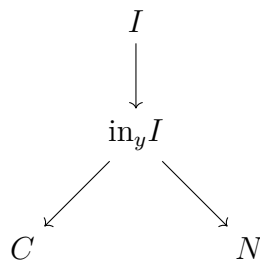
Example: $I = \langle ry, rz, z(yx - s^2) \rangle$ is not F -split.

I has a GVD at y that satisfies all the hypotheses of the theorem except that one.

III. GVD & F-splittings (Deformations)

!! Gröbner deformations can spoil F -split property !!

Definition 1.17. A property P is G -deforming if: $\text{in}_w I$ has $P \Rightarrow I$ has P .



- F -split is not (deforming)/ G -deforming
- F -rationality is deforming/ G -deforming

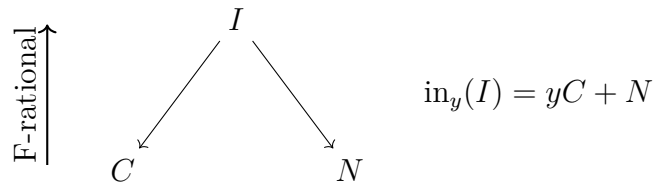
III. GVD & F-rationality

Theorem 1.18 (Bath, Walther '24). *Several broad classes of polynomials associated to matroids and satisfying Deletion-Contraction identities are F-rational.*

The Deletion-Contraction of matroids used by Bath and Walther are “the same” as the Deletion-Link in our setting.

Theorem 1.19 (Conca et al., '25). *Let $f = Lg + h \in S$, with L linear form, g, h squarefree supported polynomials (hence, F-rational), g does not divide h . Then (f) is F-rational.*

Can this be extended to GVD ideals?



Remark 1.20. Because F -rationality is G -deforming, verifying that C and N are F -rational could lift to show that the initial ideal $\text{in}_y(I) = yC + N$ and consequently I itself is F -rational.

III. GVD & F-rationality (Conditions)

Let $R = S/I$ (graded setting).

Theorem 1.21 (Fedder-Watanabe). *R is F-rational if and only if*

1. R is Cohen-Macaulay
2. $a(R) := \max\{n \mid H_m^d(R) \neq 0\} < 0$
3. R is F -injective
4. R_P is F -rational for $P \neq m$

Applying these conditions to the Geometric Vertex Decomposition (GVD) context:

1. If I is GVD then R is CM [Klein, Rajchgot]
2. $a(R) < 0$ under mild hypotheses on C [Nguyen, Rajchgot, Van Tuyl]

3. If C, N are F -rational, one can prove that $\text{in}_{<} I$ is strongly F -injective, so I is F -injective
4. ...??...

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