PASCA 2.0

Summer School at CIMAT

CIMAT

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Lecture Notes

Mutation Semigroup Algebras

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1 Cluster Algebras

Definition 1.1 (Seed)**.** Let $\Bbbk \subseteq \mathbb{K}$ be a field extension with

$$\mathrm{trdeg}_{\Bbbk}(\mathbb{K}) = n.$$

A *seed* is a pair

$$\big(\{x_1, \dots, x_n\}, B\big),$$

where $x_1, \dots, x_n \in \mathbb{K}$ are algebraically independent over $\Bbbk$, and

$$B = (b_{ij})_{1 \le i,j \le n}$$

is an $n \times n$ skew-symmetric integer matrix. The elements $x_1, \dots, x_n$ are called the *cluster variables* of the seed, and B is called the *exchange matrix*.

Definition 1.2 (Cluster mutation)**.** Let

$$\Sigma = \big(\{x_1, \dots, x_n\}, B\big)$$

be a seed, and fix an index $j \in \{1, \dots, n\}$. The *mutation of* Σ *in direction* j is the seed

$$\mu_j(\Sigma) = \big(\{x_1, \dots, x_{j-1}, x_j', x_{j+1}, \dots, x_n\}, B'\big),$$

where the new cluster variable x_j' is determined by the exchange relation

$$x_j' x_j = \prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}}.$$

The mutated exchange matrix $B' = (b'_{ik})$ is given by

$$b'_{ik} = \begin{cases} -b_{ik}, & \text{if } i = j \text{ or } k = j, \\ b_{ik} - b_{ij} b_{jk}, & \text{if } i \neq j,\ k \neq j,\ b_{ij} < 0,\ b_{jk} < 0, \\ b_{ik} + b_{ij} b_{jk}, & \text{if } i \neq j,\ k \neq j,\ b_{ij} > 0,\ b_{jk} > 0, \\ b_{ik}, & \text{otherwise.} \end{cases}$$

Equivalently, for $i, k \neq j$, one may write

$$b'_{ik} = b_{ik} + [b_{ij}]_+ [b_{jk}]_+ - [-b_{ij}]_+ [-b_{jk}]_+,$$

where $[a]_+ = \max\{a, 0\}$.

Definition 1.3 (Cluster algebra)**.** The *cluster algebra* associated to an initial seed

$$\Sigma_0 = \big(\{x_1, \dots, x_n\}, B\big)$$

is the $\Bbbk$-subalgebra

$$\mathcal{A}(\Sigma_0) \subseteq \mathbb{K}$$

generated by all cluster variables appearing in all seeds obtained from Σ_0 by iterated cluster mutations.

Example 1.4 (Rank two example)**.** Consider the initial seed

$$\left(\{x_1, x_2\}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

The exchange relations take the form

$$x_{n+2}x_n = 1 + x_{n+1}.$$

Thus

$$\begin{aligned} x_3 &= \frac{1+x_2}{x_1}, \\ x_4 &= \frac{1+x_3}{x_2} = \frac{1+x_1+x_2}{x_1x_2}, \\ x_5 &= \frac{1+x_4}{x_3}, \\ x_6 &= x_1, \qquad x_7 = x_2. \end{aligned}$$

In this example, the sequence of cluster variables is periodic of period five. The cluster algebra satisfies

$$R \subseteq \Bbbk(x_1, x_2),$$

and, by the Laurent phenomenon,

$$R \subseteq \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}].$$

The mutation pattern may be represented schematically as

$$(x_1, x_2) \xrightarrow{\mu_1} (x_3, x_2) \xrightarrow{\mu_2} (x_3, x_4) \xrightarrow{\mu_1} (x_5, x_4) \xrightarrow{\mu_2} (x_5, x_1) \xrightarrow{\mu_1} (x_2, x_1).$$

Theorem 1.5 (Laurent phenomenon)**.** *Let R be a cluster algebra with a seed*

$$\big(\{x_1,\dots,x_n\},B\big).$$

Then

$$R \subseteq \Bbbk[x_1^{\pm 1},\dots,x_n^{\pm 1}].$$

In other words, every cluster variable obtained from the initial seed by iterated mutation is a Laurent polynomial in the initial cluster variables.

2 Geometry of Cluster Algebras

Let

$$X = \operatorname{Spec}(R),$$

where R is a finitely generated cluster algebra over $\Bbbk$. A choice of seed gives an open torus chart

$$\mathbb{G}_m^n = \operatorname{Spec}\Bbbk[x_1^{\pm 1},\dots,x_n^{\pm 1}] \xhookrightarrow{j} X.$$

Equivalently, the Laurent phenomenon gives an inclusion

$$R \subseteq \Bbbk[x_1^{\pm 1},\dots,x_n^{\pm 1}],$$

and hence a dominant open morphism from the algebraic torus to X.

$$\mathbb{G}_m^n = \operatorname{Spec}\Bbbk[x_1^{\pm 1},\dots,x_n^{\pm 1}] \xhookrightarrow{\quad j\quad} X = \operatorname{Spec}(R).$$

Theorem 2.1 (Fock–Goncharov, 2008)**.** *The open torus embeddings*

$$j : \mathbb{G}_m^n \hookrightarrow X$$

arising from cluster charts cover X up to a closed subset of codimension at least 2.

Notation 2.2. On the algebraic torus $\mathbb{G}_m^n$, we denote the standard logarithmic volume form by

$$\Omega_n = \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdots x_n} = d\log x_1 \wedge \cdots \wedge d\log x_n.$$

Fact 2.3. Let

$$j_1, j_2 : \mathbb{G}_m^n \hookrightarrow X$$

be two torus charts coming from seeds of the same cluster algebra. Then the transition

birational map
$$j_1 \circ j_2^{-1} : \mathbb{G}_m^n \dashrightarrow \mathbb{G}_m^n$$
preserves the logarithmic volume form up to sign:
$$(j_1 \circ j_2^{-1})^* \Omega_n = \pm \Omega_n.$$

Remark 2.4. The sign appears because mutations act by birational transformations whose Jacobian determinant is a Laurent monomial times ± 1 in logarithmic coordinates. Thus the logarithmic differential form is preserved up to orientation.

3 Semigroup Algebras and Toric Varieties

Semigroup algebras

Let S be a commutative semigroup. The associated *semigroup algebra* is
$$\Bbbk[S] = \left\{ \sum_{u \in S} \alpha_u \chi^u \;\middle|\; \alpha_u \in \Bbbk,\ \alpha_u = 0 \text{ for all but finitely many } u \right\}.$$
Multiplication is defined by
$$\chi^u \cdot \chi^v = \chi^{u+v}.$$
Thus multiplication in $\Bbbk[S]$ is induced by addition in the semigroup S.

Proposition 3.1 (Affine toric varieties and semigroup algebras)**.** *An affine variety X is toric if and only if*
$$X \simeq \operatorname{Spec} \Bbbk[S]$$
for some finitely generated semigroup S.

Remark 3.2. The normality condition is reflected in the semigroup. More precisely,
$$X = \operatorname{Spec} \Bbbk[S] \text{ is normal} \iff S \text{ is saturated.}$$
In the standard toric notation, if N is a lattice, $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual lattice, and $\sigma \subseteq N_{\mathbb{Q}}$ is a rational polyhedral cone, then a normal affine toric variety is of the form
$$U_\sigma = \operatorname{Spec} \Bbbk[\sigma^\vee \cap M].$$
Here
$$\sigma^\vee = \{ m \in M_{\mathbb{Q}} \mid \langle u, m \rangle \geq 0 \text{ for all } u \in \sigma \}.$$

Geometric characterization

Geometrically, a variety X is toric if there is an open embedding

$$\mathbb{G}_m^n \hookrightarrow X$$

such that the action of $\mathbb{G}_m^n$ on itself extends to an action of $\mathbb{G}_m^n$ on all of X.

$$\begin{array}{ccc} \mathbb{G}_m^n \times \mathbb{G}_m^n & \longrightarrow & \mathbb{G}_m^n \\ \big\downarrow & & \big\downarrow \\ \mathbb{G}_m^n \times X & \dashrightarrow & X. \end{array}$$

Theorem 3.3 (Characterization by logarithmic volume forms)**.** *Let X be a normal variety that is either projective or affine. Then X is toric if and only if there is an open embedding*

$$j : \mathbb{G}_m^n \hookrightarrow X$$

such that the logarithmic volume form Ω_n has poles along every irreducible component of

$$X \setminus \mathbb{G}_m^n.$$

Example 3.4 (Basic toric chart)**.** For the affine plane

$$\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y],$$

the open torus is

$$\mathbb{G}_m^2 = \operatorname{Spec} \Bbbk[x^{\pm 1}, y^{\pm 1}].$$

The logarithmic volume form is

$$\Omega_2 = \frac{dx \wedge dy}{xy}.$$

It has simple poles along the boundary divisors

$$\{x = 0\} \qquad \text{and} \qquad \{y = 0\}.$$

4 Algebraic Mutation Data

Motivation

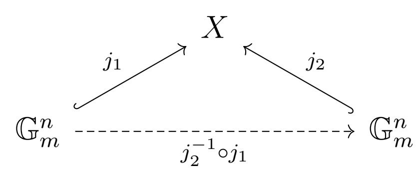
The goal is to introduce a class of algebras that contains both semigroup algebras and cluster algebras. Geometrically, one starts with an affine variety X together with torus embeddings

$$\mathbb{G}_m^n \xhookrightarrow{j_i} X.$$

The transition maps

$$j_1 \circ j_2^{-1} : \mathbb{G}_m^n \dashrightarrow \mathbb{G}_m^n$$

are required to preserve the logarithmic volume form Ω_n up to sign. Moreover, X is allowed to have poles of Ω_n along its boundary, but no zeros.



Lattice notation

Notation 4.1. Let N be a free finitely generated abelian group. Let

$$M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$$

be the dual lattice. We write

$$N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The natural pairing is denoted

$$\langle -, - \rangle : N \times M \longrightarrow \mathbb{Z}.$$

Definition 4.2 (Algebraic mutation datum)**.** Let $\sigma \subseteq N_{\mathbb{Q}}$ be a rational polyhedral cone. An *algebraic mutation datum* is a pair

$$(u, h),$$

where

$$u \in N, \qquad h = g^k, \qquad g \in \mathbb{k}[\sigma^\vee \cap M],$$

for some integer $k \geq 1$. We say that (u, h) is σ-*admissible* if

$$u \notin \sigma.$$

Remark 4.3. For a cone $\sigma \subseteq N_{\mathbb{Q}}$, the associated affine toric variety is

$$U_\sigma = \operatorname{Spec} \Bbbk[\sigma^\vee \cap M].$$

Thus the condition

$$g \in \Bbbk[\sigma^\vee \cap M]$$

means that g is a regular function on the affine toric chart U_σ.

Remark 4.4. The condition $u \notin \sigma$ is a nontriviality condition: it prevents the mutation direction from being entirely contained in the cone defining the toric chart. In geometric terms, the mutation changes the boundary data of the toric chart rather than merely acting inside the same toric monoid.

Definition 4.5 (Algebraic mutation)**.** Let

$$U_{\sigma_1} = \operatorname{Spec} \Bbbk[\sigma_1^\vee \cap M], \qquad U_{\sigma_2} = \operatorname{Spec} \Bbbk[\sigma_2^\vee \cap M]$$

be affine toric varieties with common character lattice M. A birational map

$$\mu : U_{\sigma_1} \dashrightarrow U_{\sigma_2}$$

is called an *algebraic mutation* if the induced pullback on the common function field

$$\mu^* : \Bbbk(M) \longrightarrow \Bbbk(M)$$

is given on characters by

$$\mu^*(\chi^m) = \chi^m h^{-\langle u,m\rangle}$$

for some σ_2-admissible algebraic mutation datum (u, h).

Remark 4.6. An algebraic mutation datum (u, h) and the corresponding mutation μ induce a bijection between the torus-invariant divisors of the two toric models. This is one reason algebraic mutations preserve much of the codimension-one geometry.

$$U_{\sigma_1} \overset{\mu}{\dashrightarrow} U_{\sigma_2}$$

$$\operatorname{Spec} \Bbbk[\sigma_1^\vee \cap M] \dashrightarrow \operatorname{Spec} \Bbbk[\sigma_2^\vee \cap M]$$

5 Mutation Semigroup Algebras

Definition 5.1 (Mutation semigroup algebra)**.** A *mutation semigroup algebra*, abbreviated *MSA*, is a finitely generated $\mathbb{k}$-algebra R that can be written as a finite intersection

$$R = R_0 \cap R_1 \cap \cdots \cap R_s$$

inside a common field, satisfying the following conditions. For each $i \in \{0, \ldots, s\}$:

(1) There is an embedding

$$j_i : \mathbb{k}[\sigma_i^\vee \cap M] \hookrightarrow \mathrm{Frac}(R)$$

such that

$$R_i = \mathrm{Im}(j_i).$$

Equivalently, each R_i is an affine toric semigroup algebra realized inside the common fraction field of R.

(2) The birational transition map

$$j_0 \circ j_i^{-1}$$

is an algebraic mutation.

(3) If $P \subseteq R_i$ is a prime ideal of height one, then

$$P \cap R \subseteq R$$

is a prime ideal of height one.

Remark 5.2. Condition (3) ensures that codimension-one information does not collapse when passing from an individual toric chart R_i to the intersection ring R. This is essential if one wants to control divisors, canonical classes, singularities, or symbolic powers through the toric charts.

Remark 5.3. The inclusions of classes discussed in the lecture are

$$\{\text{semigroup algebras}\} \subseteq \{\text{mutation semigroup algebras}\}$$

and

$$\{\text{cluster algebras}\} \subseteq \{\text{mutation semigroup algebras}\}.$$

Thus mutation semigroup algebras are designed to envelop both toric semigroup algebras and cluster algebras.

Example 5.4. Consider the intersection

$$\Bbbk[x,y] \cap \Bbbk\left[\frac{y+1}{x}, y\right]$$

inside the common field $\Bbbk(x,y)$. If we set

$$z = \frac{y+1}{x},$$

then

$$xz = y + 1.$$

Therefore

$$\Bbbk[x,y] \cap \Bbbk\left[\frac{y+1}{x}, y\right] \cong \Bbbk[x,y,z]/(xz - y - 1).$$

This gives a basic example of a mutation semigroup algebra arising from a single algebraic mutation.

6 Guiding Questions

The lecture raised the following questions.

Question 6.1. Which mutation semigroup algebras admit geometric vertex decompositions?

Question 6.2. Are mutation semigroup algebras F-pure? Are they strongly F-regular?

Question 6.3. How do symbolic powers behave for mutation semigroup algebras? In particular, what can be said about Harbourne-type containment problems for mutation semigroup algebras?

Remark 6.4. The symbolic power question is natural because mutation semigroup algebras are built from toric charts, and toric geometry often gives strong control over divisors and valuations. However, the intersection of several mutated toric charts may introduce new codimension-two behavior, so containment problems can be substantially subtler than in the purely toric case.

7 Fano Varieties and Compactifications

Definition 7.1 (Fano variety)**.** A *Fano variety* is a normal projective variety X such that the anti-canonical divisor

$$-K_X$$

is ample.

Theorem 7.2 (EFMS25)**.** *Let R be a mutation semigroup algebra. If*

$$U = \operatorname{Spec} R$$

has klt singularities, then there exists an embedding

$$U \hookrightarrow X,$$

where X is a projective klt Fano variety.

Remark 7.3. The phrase "U has klt singularities" means that the affine variety $U = \operatorname{Spec} R$ has Kawamata log terminal singularities. The theorem therefore connects the local singularity theory of mutation semigroup algebras with global Fano geometry.

Theorem 7.4 (Dimension two)**.** *In dimension 2, any general smooth Fano surface is the compactification of*

$$\operatorname{Spec}(\mathrm{MSA}),$$

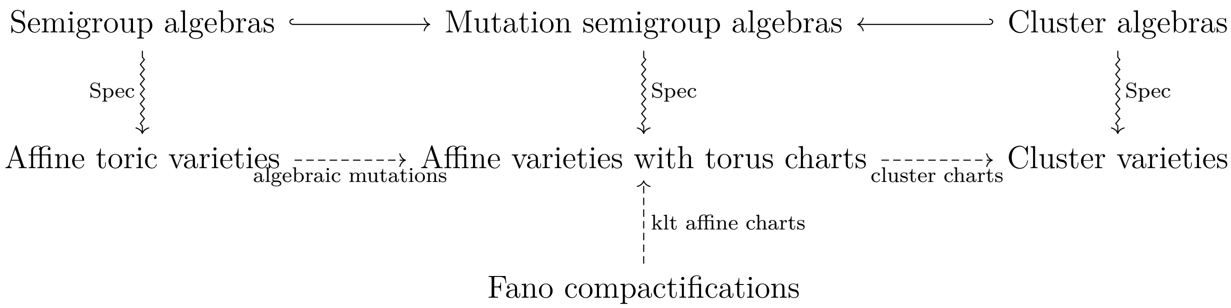
that is, of the spectrum of a mutation semigroup algebra.

Theorem 7.5 (Dimension three)**.** *In dimension 3, there are 84 families of smooth Fano threefolds that are compactifications of*

$$\operatorname{Spec}(\mathrm{MSA}).$$

8 Summary Diagram

The lecture can be summarized by the following diagram of ideas:



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