PASCA 2.0

Summer School at CIMAT

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Lecture Notes

The module of derivations of graded Tjurina algebras

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1 Lecture 1- May 25, 2026

The Tjurina Algebra

Let

$$S = \mathbb{C}[x_1, \ldots, x_n].$$

For a polynomial $f \in S$, the *Jacobian ideal* of f is

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle.$$

Definition 1.1 (Tjurina algebra)**.** The *Tjurina algebra* of f is

$$T(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f, J_f \rangle}.$$

The number

$$\tau(f) := \dim_{\mathbb{C}} T(f)$$

is called the *Tjurina number*.

Theorem 1.2 (Mather–Yau)**.** *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ define isolated hypersurface singularity germs at the origin. Then the germs*

$$(V(f), 0) \quad \text{and} \quad (V(g), 0)$$

are analytically isomorphic if and only if their local Tjurina algebras

$$\frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\langle f, J_f \rangle} \quad \text{and} \quad \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\langle g, J_g \rangle}$$

are isomorphic as local $\mathbb{C}$-algebras.

Remark 1.3. The Mather–Yau theorem says that the analytic isomorphism type of an isolated hypersurface singularity is completely determined by its Tjurina algebra. Thus the finite-dimensional algebra $T(f)$ packages the local analytic information of the singularity.

Yau's Lie Algebra

Definition 1.4 (Yau algebra)**.** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ define an isolated hypersurface singularity. The *Yau algebra* associated to f is the Lie algebra

$$L(f) := \mathrm{Der}_{\mathbb{C}}(T(f)).$$

Theorem 1.5 (Yau, 1981)**.** *Let f define an isolated hypersurface singularity. Then*

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f))$$

is a finite-dimensional solvable Lie algebra.

Program of Questions

The lecture notes outline the following sequence of questions.

(1) Does the Lie algebra

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f))$$

distinguish isolated hypersurface singularities?

(2) If $f \in \mathbb{C}[\underline{x}]$ is weighted homogeneous, then $T(f)$ is graded and hence

$$L(f) = \bigoplus_{j \in \mathbb{Z}} L_j$$

is a graded Lie algebra.

(3) Yau's conjectural direction concerns the negative-degree part of this graded Lie algebra. In the notation of the notes, this appears as

$$L_j \neq 0 \quad \text{for certain } j < 0.$$

(4) If f is weighted homogeneous, then by Milnor–Orlik the dimension of the Tjurina algebra depends only on the weights:

$$\dim_{\mathbb{C}} T(f) \quad \text{depends only on the weights of } f.$$

(5) Yau–Zuo type conjectures concern bounds for

$$\dim_{\mathbb{C}} \mathrm{Der}_{\mathbb{C}}(L(f)).$$

(6) Higher-order versions of these invariants and questions can be formulated.

Complex Hypersurfaces

Basic Definitions

Let

$$f \in \mathbb{C}[x_1, \dots, x_n].$$

The complex hypersurface defined by f is

$$V(f) = \{p \in \mathbb{C}^n \mid f(p) = 0\} \subsetneq \mathbb{C}^n.$$

Definition 1.6 (Singular locus)**.** The singular locus of the hypersurface $V(f)$ is

$$\mathrm{Sing}(f) = \left\{ p \in \mathbb{C}^n \;\middle|\; f(p) = 0 \text{ and } \frac{\partial f}{\partial x_i}(p) = 0 \text{ for all } i = 1, \dots, n \right\}.$$

Equivalently,

$$\mathrm{Sing}(f) = V(f, J_f).$$

Definition 1.7 (Isolated hypersurface singularity)**.** The hypersurface $V(f)$ has an *isolated hypersurface singularity* at the origin if

$$\mathrm{Sing}(f) = \{\underline{0}\}$$

in a sufficiently small analytic neighborhood of the origin.

Examples

Example 1.8 (The cusp)**.** The plane curve

$$V(x^3 - y^2) \subseteq \mathbb{C}^2$$

has a cusp singularity at the origin.

Example 1.9 (The quadratic cone)**.** The hypersurface

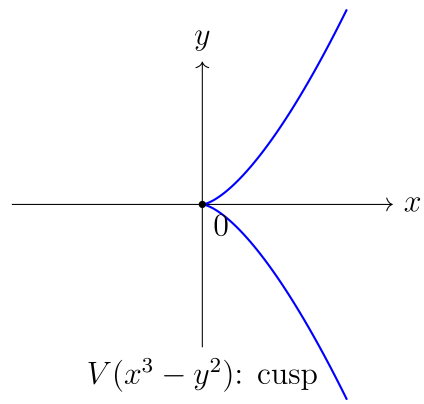
$$V(x^2 - y^2 - z^2) \subseteq \mathbb{C}^3$$

is a cone with an isolated singularity at the origin.

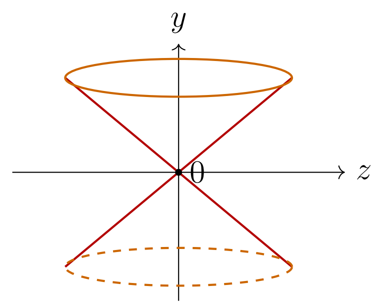
Example 1.10 (The Whitney umbrella)**.** The surface

$$V(x^2z - y^2) \subseteq \mathbb{C}^3$$

is the Whitney umbrella.



$V(x^3 - y^2)$: cusp



$V(x^2 - y^2 - z^2)$: quadratic cone
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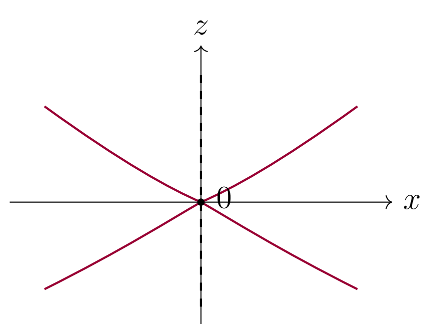

$V(x^2z - y^2)$: Whitney umbrella
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Milnor and Tjurina Algebras

Definition 1.11 (Milnor algebra)**.** The *Milnor algebra* of f is

$$M(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{J_f} = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}.$$

Its complex vector space dimension

$$\mu(f) := \dim_{\mathbb{C}} M(f)$$

is called the *Milnor number*.

Definition 1.12 (Tjurina algebra)**.** The *Tjurina algebra* of f is

$$T(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f, J_f \rangle} = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}$$

Its complex vector space dimension

$$\tau(f) := \dim_{\mathbb{C}} T(f)$$

is called the *Tjurina number*.

Remark 1.13. The Milnor number $\mu(f)$ measures the topology of the Milnor fiber. For an isolated hypersurface singularity, the Milnor fiber has the homotopy type of a bouquet of $\mu(f)$ spheres. The Tjurina number $\tau(f)$ measures the dimension of the base of the semi-universal deformation. If f is weighted homogeneous of weighted degree d with weights $w_1, \ldots, w_n$, and if f has an isolated singularity at the origin, then

$$\mu(f) = \tau(f).$$

Moreover, the Milnor number is determined by the weight system and the weighted degree. Equivalently, after normalizing the weights by d, it depends only on the normalized weights.

Derivations

Derivations of Algebras

Definition 1.14 (Derivation)**.** Let R be a k-algebra. A k-derivation of R is a k-linear map

$$D : R \to R$$

satisfying the Leibniz rule

$$D(ab) = aD(b) + bD(a)$$

for all $a, b \in R$. The set of all k-derivations of R is denoted by

$$\mathrm{Der}_k(R).$$

Example 1.15. Let

$$R = \mathbb{C}[x_1, \dots, x_n].$$

Then each partial derivative

$$\partial_{x_i} : R \to R$$

is a $\mathbb{C}$-derivation. Moreover, every $\mathbb{C}$-derivation $D : R \to R$ can be written uniquely as

$$D = \sum_{i=1}^{n} D(x_i)\partial_{x_i}.$$

Therefore

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n]) \cong \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \dots, x_n]\partial_{x_i}$$

is a free R-module of rank n.

Derivations on Quotients

Question 1.16. How can one describe

$$\mathrm{Der}_{\mathbb{C}}\left(\frac{\mathbb{C}[x_1, \dots, x_n]}{I}\right)?$$

A naive attempt is to descend the usual partial derivatives to the quotient. This does not always work.

Example 1.17. Let

$$A = \frac{\mathbb{C}[x]}{\langle x^2 \rangle}.$$

The usual derivative

$$\partial_x : \mathbb{C}[x] \to \mathbb{C}[x]$$

does not induce a well-defined derivation on A. Indeed,

$$[x^2] = [0] \in A,$$

but

$$\partial_x(x^2) = 2x,$$

and

$$[2x] \neq [0] \in A.$$

Equivalently,

$$\partial_x(\langle x^2 \rangle) \nsubseteq \langle x^2 \rangle.$$

Definition 1.18 (Derivations preserving an ideal)**.** Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Define

$$\operatorname{Der}_I(\mathbb{C}[x_1, \ldots, x_n]) = \{D \in \operatorname{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]) \mid D(I) \subseteq I\}.$$

These are the derivations preserving the ideal I.

Proposition 1.19. *Let*

$$S = \mathbb{C}[x_1, \ldots, x_n]$$

and let $I \subseteq S$ be an ideal. Then there is a canonical isomorphism

$$\operatorname{Der}_{\mathbb{C}}\left(\frac{S}{I}\right) \cong \frac{\operatorname{Der}_I(S)}{I \operatorname{Der}_{\mathbb{C}}(S)}.$$

Proof. Every derivation $D \in \operatorname{Der}_I(S)$ induces a derivation

$$\overline{D} : S/I \to S/I$$

by

$$\overline{D}([f]) = [D(f)].$$

This is well-defined precisely because $D(I) \subseteq I$.

Thus we obtain a natural map

$$\operatorname{Der}_I(S) \longrightarrow \operatorname{Der}_{\mathbb{C}}(S/I).$$

Its kernel consists of those derivations D such that

$$D(S) \subseteq I.$$

Since

$$\mathrm{Der}_{\mathbb{C}}(S) = \bigoplus_{i=1}^{n} S\partial_{x_i},$$

this kernel is exactly

$$I\,\mathrm{Der}_{\mathbb{C}}(S).$$

Therefore the induced map gives the desired isomorphism

$$\frac{\mathrm{Der}_I(S)}{I\,\mathrm{Der}_{\mathbb{C}}(S)} \cong \mathrm{Der}_{\mathbb{C}}(S/I).$$

□

$$\begin{array}{ccc} S & \xrightarrow{D} & S \\ \downarrow & & \downarrow \\ S/I & \xrightarrow[\overline{D}]{} & S/I \end{array}$$

Lie Algebras

Basic Definitions

Definition 1.20 (Lie algebra)**.** A Lie algebra over $\mathbb{C}$ is a $\mathbb{C}$-vector space L together with a bilinear map

$$[-,-] : L \times L \to L, \qquad (X,Y) \mapsto [X,Y],$$

called the *Lie bracket*, satisfying:

i) $[X,X] = 0$ for all $X \in L$.

ii) The Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in L$.

Remark 1.21. Since the base field is $\mathbb{C}$, the condition $[X,X] = 0$ is equivalent to skew-symmetry:

$$[X,Y] = -[Y,X].$$

Example 1.22 (Derivations form a Lie algebra)**.** Let A be a $\mathbb{C}$-algebra. Then

$$\mathrm{Der}_{\mathbb{C}}(A)$$

is a Lie algebra with bracket

$$[D, E] = D \circ E - E \circ D.$$

Definition 1.23 (Homomorphism of Lie algebras)**.** Let L_1 and L_2 be Lie algebras. A map

$$\Phi : L_1 \to L_2$$

is a homomorphism of Lie algebras if:

i) Φ is $\mathbb{C}$-linear.

ii) For all $X, Y \in L_1$,

$$\Phi([X, Y]_{L_1}) = [\Phi(X), \Phi(Y)]_{L_2}.$$

If Φ is bijective, then Φ is an isomorphism of Lie algebras.

Derivation Lie Algebras of Tjurina Algebras

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ define an isolated hypersurface singularity. The central algebraic object in these notes is

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f)).$$

Since

$$T(f) = \frac{S}{\langle f, J_f \rangle},$$

the quotient description of derivations gives

$$L(f) \cong \frac{\mathrm{Der}_{\langle f, J_f \rangle}(S)}{\langle f, J_f \rangle \, \mathrm{Der}_{\mathbb{C}}(S)}.$$

Remark 1.24. This formula allows one to compute $L(f)$ explicitly. A derivation

$$D = \sum_{i=1}^{n} a_i \partial_{x_i}$$

descends to $T(f)$ precisely when it preserves the Tjurina ideal:

$$D(\langle f, J_f \rangle) \subseteq \langle f, J_f \rangle.$$

Weighted Homogeneous Case

Definition 1.25 (Weighted homogeneous polynomial)**.** Let $w_1, \ldots, w_n$ be positive rational weights. A polynomial

$$f \in \mathbb{C}[x_1, \ldots, x_n]$$

is weighted homogeneous of weighted degree d if each monomial

$$x_1^{a_1} \cdots x_n^{a_n}$$

appearing in f satisfies

$$a_1 w_1 + \cdots + a_n w_n = d.$$

Remark 1.26. If f is weighted homogeneous, then the Euler derivation

$$E = \sum_{i=1}^{n} w_i x_i \partial_{x_i}$$

satisfies the Euler identity

$$E(f) = df.$$

Consequently,

$$f \in J_f$$

whenever the weighted degree d is nonzero in the base field. Over $\mathbb{C}$, this always holds. Thus, in the weighted homogeneous isolated hypersurface case,

$$T(f) = M(f).$$

After clearing denominators, we may assume the weights $w_1, \ldots, w_n$ are positive integers. Then S, $T(f)$, and

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f))$$

inherit $\mathbb{Z}$-gradings:

$$L(f) = \bigoplus_{j \in \mathbb{Z}} L_j.$$

Absolutely Isolated Surface Singularities

Double Points

Definition 1.27 (Double point)**.** Let

$$f = f_2 + f_3 + f_4 + \cdots$$

be the decomposition of a convergent power series into homogeneous parts, and assume $f(0) = 0$ and $df(0) = 0$. If

$$f_2 \neq 0,$$

then the hypersurface singularity $V(f)$ at the origin has multiplicity 2 and is called a double point.

Definition 1.28 (Absolutely isolated singularity)**.** Let

$$X = V(f) \subseteq \mathbb{C}^n$$

and suppose $0 \in X$ is an isolated singularity. The singularity is called *absolutely isolated* if a resolution of singularities of X can be obtained by blowing up only points.

Example 1.29. The surface

$$V(x^2 - y^2 - z^2)$$

is an absolutely isolated singularity.

Example 1.30 (The Whitney umbrella)**.** The surface

$$V(x^2z - y^2) \subseteq \mathbb{C}^3$$

is the Whitney umbrella. Its singular locus is the line

$$\{x = 0, \ y = 0\}.$$

Thus it is not an isolated hypersurface singularity.

ADE Classification

Theorem 1.31 (ADE classification of rational double points)**.** *Let*

$$X = V(f) \subseteq \mathbb{C}^3$$

be a normal surface singularity at the origin. Then X is a rational double point, equivalently a Du Val singularity, if and only if it is analytically isomorphic to exactly one of the following hypersurface singularities:

$$A_k: \quad x^{k+1} + y^2 + z^2 = 0, \qquad k \geq 1,$$

$$D_k: \quad x^{k-1} + xy^2 + z^2 = 0, \qquad k \geq 4,$$

$$E_6: \quad x^3 + y^4 + z^2 = 0,$$

$$E_7: \quad x^3 + xy^3 + z^2 = 0,$$

$$E_8: \quad x^3 + y^5 + z^2 = 0.$$

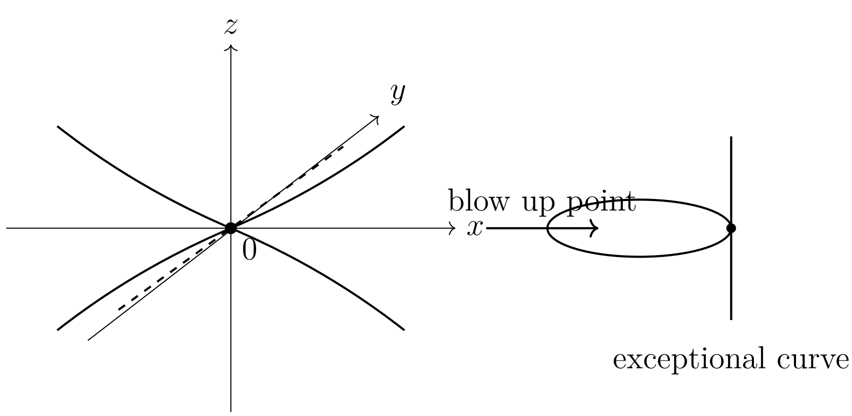
Remark 1.32. The ADE singularities are also called simple surface singularities or rational double points. Their classification is governed by the simply-laced Dynkin diagrams

$$A_k, \quad D_k, \quad E_6, \quad E_7, \quad E_8.$$

These diagrams also appear as the dual intersection graphs of exceptional divisors in the minimal resolution.

Schematic Picture

The following schematic diagram represents the idea that a double point surface singularity has a distinguished singular point at the origin, and resolution may proceed by blowing up points in the absolutely isolated case.



Summary of the Lecture

The notes establish the following chain of ideas.

(1) A hypersurface singularity

$$V(f) \subseteq \mathbb{C}^n$$

has an associated Jacobian ideal J_f.

(2) The Tjurina algebra

$$T(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f, J_f \rangle}$$

is finite-dimensional when f defines an isolated hypersurface singularity.

(3) By the Mather–Yau theorem, the Tjurina algebra determines the analytic isomorphism class of the isolated hypersurface singularity.

(4) The derivation Lie algebra

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f))$$

is a secondary invariant attached to the singularity.

(5) For weighted homogeneous singularities, $L(f)$ inherits a grading

$$L(f) = \bigoplus_j L_j.$$

(6) In surface singularity theory, absolutely isolated double points are classified by the ADE list:

$$A_k, \quad D_k, \quad E_6, \quad E_7, \quad E_8.$$

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Let $R = \mathbb{C}[x_1, \ldots, x_n]$ or $R = \mathbb{C}\{x_1, \ldots, x_n\}$. Let $f \in R$ and $X = \{f = 0\}$.

Definition 2.1 (Jacobian ideal)**.** The Jacobian ideal of f is

$$J_f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \subseteq R.$$

Definition 2.2 (Tjurina algebra)**.** The Tjurina algebra of the hypersurface singularity X is

$$T(X) = T(f) = \frac{R}{(f, J_f)} = \frac{R}{\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}.$$

Definition 2.3 (Yau algebra)**.** The Lie algebra associated to X, often called the Yau algebra in this context, is

$$\mathcal{L}(X) = \mathrm{Der}_{\mathbb{C}}(T(X)).$$

Recall that a derivation is a linear map $D : T(X) \longrightarrow T(X)$ satisfying the Leibniz rule:

$$D(ab) = aD(b) + bD(a).$$

The Lie bracket is given by the commutator:

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

Remark 2.4 (Added Context)**.** If $T(X) = R/I$, then a derivation D' can be lifted to a derivation D satisfying

$$D(I) \subseteq I.$$

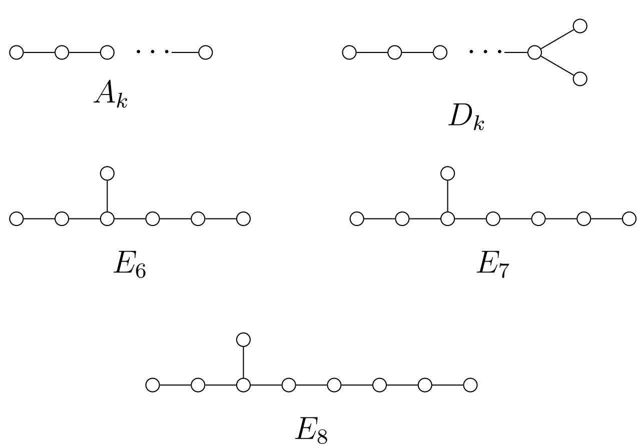
This descends to a derivation of $T(X)$. This is the basic mechanism used below to compute $\mathcal{L}(A_k)$.

ADE Surface Singularities

The ADE surface singularities in $\mathbb{C}^3$ are the following hypersurfaces.

$$\begin{aligned}
A_k: &\quad x^{k+1} + y^2 + z^2 = 0, \qquad k \geq 1,\\
D_k: &\quad x^{k-1} + xy^2 + z^2 = 0, \qquad k \geq 4,\\
E_6: &\quad x^3 + y^4 + z^2 = 0,\\
E_7: &\quad x^3 + xy^3 + z^2 = 0,\\
E_8: &\quad x^3 + y^5 + z^2 = 0.
\end{aligned}$$

These are pairwise non-isomorphic hypersurface singularities.



Theorem 2.5 (Elashvili–Khimshiashvili)**.** *Let X and Y be ADE singularities. Except for the exceptional pair*

$$\{X, Y\} = \{A_6, D_5\},$$

we have

$$X \simeq Y \quad \Longleftrightarrow \quad \mathcal{L}(X) \simeq \mathcal{L}(Y)$$

Remark 2.6. The exceptional phenomenon is

$$A_6 \not\simeq D_5, \qquad \mathcal{L}(A_6) \simeq \mathcal{L}(D_5).$$

This means the Yau algebra distinguishes almost all ADE singularities, but not quite.

Computation of $\mathcal{L}(A_k)$

Let

$$A_k = \{x^{k+1} + y^2 + z^2 = 0\}.$$

We have $f = x^{k+1} + y^2 + z^2$. The Jacobian ideal is

$$J_f = (x^k, y, z),$$

because

$$\frac{\partial f}{\partial x} = (k+1)x^k, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial z} = 2z.$$

Hence the Tjurina algebra is

$$T(A_k) = \frac{\mathbb{C}[x, y, z]}{(x^{k+1} + y^2 + z^2, x^k, y, z)} \cong \frac{\mathbb{C}[x]}{(x^k)}.$$

A basis is given by $\{\overline{1}, \overline{x}, \ldots, \overline{x^{k-1}}\}$.

Proposition 2.7. *There is a $\mathbb{C}$-basis*

$$\mathcal{L}(A_k) = \operatorname{span}_{\mathbb{C}} \left\{ \overline{x\partial_x}, \overline{x^2\partial_x}, \ldots, \overline{x^{k-1}\partial_x} \right\}.$$

In particular, $\dim_{\mathbb{C}} \mathcal{L}(A_k) = k - 1$.

Proof. Let

$$A = \frac{\mathbb{C}[x]}{(x^k)}.$$

A derivation $\overline{D} : A \to A$ lifts to $D' : \mathbb{C}[x] \to \mathbb{C}[x]$ such that

$$D'\Big((x^k)\Big) \subseteq (x^k).$$

We can write $D'(g) = \frac{\partial g}{\partial x} D'(x)$. Write

$$D'(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{k-1} x^{k-1} \qquad \text{modulo } (x^k),$$

with the condition $D'(x^k) \in (x^k)$. We have

$$D'(x^k) = kx^{k-1} D'(x).$$

Modulo (x^k), this becomes

$$kx^{k-1} D'(x) \equiv k\lambda_0 x^{k-1}.$$

For this to be zero, we need $k\lambda_0 \overline{x}^{k-1} = 0$ in A. Thus, we get

$$\lambda_0 = 0.$$

So $D'(x) \in (x)/(x^k)$, which means the derivations are generated by

$$x\partial_x,\ x^2\partial_x,\ \ldots,\ x^{k-1}\partial_x.$$

These are linearly independent on A. Therefore they form a basis of $\mathcal{L}(A_k)$, and

$$\dim_{\mathbb{C}} \mathcal{L}(A_k) = k - 1.$$

□

Definition 2.8. For $1 \leq i \leq k-1$, set

$$e_i = x^i \partial_x \in \mathcal{L}(A_k).$$

Then $\mathcal{L}(A_k) = \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_{k-1}\}$.

Lemma 2.9. *For $1 \leq i, j \leq k-1$, the Lie bracket in $\mathcal{L}(A_k)$ is*

$$[e_i, e_j] = (j - i)e_{i+j-1},$$

with the convention $e_m = 0$ if $m \geq k$.

Proof. Using

$$e_i = x^i \partial_x, \qquad e_j = x^j \partial_x,$$

we compute

$$\begin{aligned}[x^i\partial_x, x^j\partial_x] &= x^i\partial_x(x^j)\partial_x - x^j\partial_x(x^i)\partial_x \\ &= jx^{i+j-1}\partial_x - ix^{i+j-1}\partial_x \\ &= (j-i)x^{i+j-1}\partial_x.\end{aligned}$$

Thus $[e_i, e_j] = (j-i)e_{i+j-1}$. □

Dimensions of the ADE Yau Algebras

The lecture records the following dimensions:

Singularity	$\dim_{\mathbb{C}} \mathcal{L}(\cdot)$
A_k	$k-1$
D_k	k
E_6	7
E_7	8
E_8	10

Equivalently,

$$\begin{aligned}&\dim_{\mathbb{C}} \mathcal{L}(A_k) = k-1, \quad \dim_{\mathbb{C}} \mathcal{L}(D_k) = k, \\ &\dim_{\mathbb{C}} \mathcal{L}(E_6) = 7, \qquad \dim_{\mathbb{C}} \mathcal{L}(E_7) = 8, \quad \dim_{\mathbb{C}} \mathcal{L}(E_8) = 10.\end{aligned}$$

These dimensions distinguish most ADE singularities by their associated Lie algebras. The dimension count alone leaves the following possible collisions:

$$\mathcal{L}(A_k) \quad \text{and} \quad \mathcal{L}(D_{k-1}) \qquad (k = 5 \text{ or } k \geq 7),$$

and possible collisions with exceptional singularities:

$$(D_7, E_6), \quad (D_{10}, E_8), \quad (A_8, E_6), \quad (A_{11}, E_8),$$

$$(A_9, E_7), \qquad (D_8, E_7).$$

Remark 2.10. The notes indicate that the above equal-dimensional cases are distinguished by finer Lie-theoretic invariants, such as the upper central series, Cartan rank, indecomposability, and nilpotent ideals.

Nilpotent Lie Algebras and Cartan Subalgebras

Definition 2.11 (Lower central series)**.** Let L be a finite-dimensional Lie algebra. Define

$$L^0 = L, \qquad L^1 = [L, L],$$

$$L^n = [L, L^{n-1}] = \operatorname{span}_{\mathbb{C}}\{[x, y] \mid x \in L,\ y \in L^{n-1}\} \qquad (n \geq 1).$$

We say L is nilpotent if $L^n = 0$ for some n.

Remark 2.12. Some notes use the indexing

$$L_0 = L, \qquad L_1 = [L, L], \qquad L_2 = [L, L_1], \qquad \ldots.$$

Definition 2.13 (Cartan subalgebra)**.** Let L be a finite-dimensional Lie algebra. A Cartan subalgebra of L is a nilpotent subalgebra H satisfying the self-normalizing condition

$$N_L(H) = H,$$

where

$$N_L(H) = \{y \in L \mid [x, y] \in H \text{ for all } x \in H\}.$$

That is, if $[x, y] \in H$ for all $x \in H \implies y \in H$.

Theorem 2.14. *All Cartan subalgebras of a finite-dimensional complex Lie algebra have the same dimension. This common dimension is called the rank of L, denoted*

$$\operatorname{rk}(L).$$

Cartan Rank of $\mathcal{L}(A_k)$

Recall that

$$\mathcal{L}(A_k) = \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_{k-1}\}, \qquad e_i = x^i \partial_x.$$

Let $H = \operatorname{span}_{\mathbb{C}}\{e_1\}$.

Claim 2.15. *The subspace H is a Cartan subalgebra of $\mathcal{L}(A_k)$. In particular,*

$$\operatorname{rk}(\mathcal{L}(A_k)) = 1.$$

Proof. First, H is nilpotent because

$$[H,H] = \operatorname{span}_{\mathbb{C}}\{[e_1,e_1]\} = 0.$$

Now we check the self-normalizing condition. Let

$$y = \sum_{i=1}^{k-1} \lambda_i e_i \in \mathcal{L}(A_k).$$

Since $[e_1, e_i] = (i-1)e_i$, we have

$$\begin{aligned}
[e_1, y] &= \left[e_1, \sum_{i=1}^{k-1} \lambda_i e_i\right] \\
&= \sum_{i=1}^{k-1} \lambda_i [e_1, e_i] \\
&= \sum_{i=2}^{k-1} \lambda_i (i-1) e_i.
\end{aligned}$$

If $[e_1, y] \in H = \operatorname{span}_{\mathbb{C}}\{e_1\}$, then the coefficients must vanish for $i \geq 2$. Since $i - 1 \neq 0$ for $i \geq 2$, we have

$$\lambda_i = 0 \qquad \text{for all } i \geq 2.$$

Thus $y = \lambda_1 e_1 \in H$. □

The lecture notes also record the analogous rank statements:

$$\operatorname{rk}(\mathcal{L}(D_k)) = 1,$$

$$\operatorname{rk}(\mathcal{L}(E_6)) = \operatorname{rk}(\mathcal{L}(E_8)) = 2.$$

Remark 2.16 (Added Context)**.** The notes emphasize that Cartan rank helps distinguish some Lie algebras that have the same dimension. Dimension is a coarse invariant, while Cartan rank and central-series data are finer invariants.

The Exceptional Pair A_6 and D_5

The main exceptional case is

$$A_6 \not\simeq D_5, \qquad \mathcal{L}(A_6) \simeq \mathcal{L}(D_5).$$

For A_6, we have

$$T(A_6) \cong \frac{\mathbb{C}[x]}{(x^6)}.$$

$$\mathcal{L}(A_6) = \operatorname{span}_{\mathbb{C}}\{e_1, e_2, e_3, e_4, e_5\},$$

where $e_i = x^i \partial_x$.

For D_5, the hypersurface is

$$D_5 = \{x^4 + xy^2 + z^2 = 0\}.$$

The derivations are spanned by:

$$\begin{aligned} d_1 &= 3y\partial_y + 2x\partial_x, \\ d_2 &= 4x^2\partial_y + y\partial_x, \\ d_3 &= x^3\partial_y, \\ d_4 &= x^2\partial_x, \\ d_5 &= x^3\partial_x. \end{aligned}$$

So

$$\mathcal{L}(D_5) = \operatorname{span}_{\mathbb{C}}\{d_1, d_2, d_3, d_4, d_5\}.$$

Theorem 2.17. *There is an isomorphism of Lie algebras*

$$\mathcal{L}(A_6) \cong \mathcal{L}(D_5).$$

Proof. Using the bases

$$\mathcal{L}(A_6) = \operatorname{span}_{\mathbb{C}}\{e_1, e_2, e_3, e_4, e_5\}$$

and

$$\mathcal{L}(D_5) = \operatorname{span}_{\mathbb{C}}\{d_1, d_2, d_3, d_4, d_5\},$$

define a linear map $\Phi : \mathcal{L}(A_6) \longrightarrow \mathcal{L}(D_5)$ by

$$\begin{aligned} &e_1 \longmapsto d_1, \quad e_2 \longmapsto d_2, \qquad e_3 \longmapsto -\frac{1}{8}d_3, \\ &e_4 \longmapsto d_4, \quad e_5 \longmapsto -\frac{1}{2}d_5. \end{aligned}$$

The lecture notes state that, after computing all commutators $[-,-]$ among the d_i, one has

$$\Phi([e_i, e_j]) = [\Phi(e_i), \Phi(e_j)] \qquad \text{for all } 1 \le i, j \le 5.$$

□

Remark 2.18. This proves the exceptional behavior:

$$A_6 \not\simeq D_5, \qquad \mathcal{L}(A_6) \simeq \mathcal{L}(D_5).$$

This is the single exceptional pair.

Summary of the Lecture

The lecture establishes the following chain of ideas.

(1) To a hypersurface singularity X, one attaches its Tjurina algebra

$$T(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{(f, J_f)}.$$

(2) The derivation Lie algebra

$$\mathcal{L}(X) = \mathrm{Der}_{\mathbb{C}}(T(X))$$

is finite-dimensional.

(3) For the ADE singularities, $\mathcal{L}(X)$ almost determines X.

(4) The dimensions are

X	$\dim_{\mathbb{C}} \mathcal{L}(X)$
A_k	$k-1$
D_k	k
E_6	7
E_7	8
E_8	10

(5) When dimensions coincide, one uses finer invariants such as Cartan rank, central series, indecomposability, and nilpotent ideals.

(6) The only exceptional ADE pair is

$$A_6, \ D_5,$$

where

$$A_6 \not\simeq D_5 \quad \text{but} \quad \mathcal{L}(A_6) \simeq \mathcal{L}(D_5).$$

3 Lecture 3- May 27, 2026

Definition 3.1 (Milnor algebra and Tjurina algebra)**.** The *Milnor algebra* of f is

$$M(f) = \frac{\mathbb{C}[\mathbf{x}]}{J_f}.$$

The *Tjurina algebra* of f is

$$T(f) = \frac{\mathbb{C}[\mathbf{x}]}{(f, J_f)}.$$

Since

$$J_f \subseteq (f, J_f),$$

there is a natural surjection

$$M(f) = \frac{\mathbb{C}[\mathbf{x}]}{J_f} \twoheadrightarrow \frac{\mathbb{C}[\mathbf{x}]}{(f, J_f)} = T(f).$$

Consequently,

$$\dim_{\mathbb{C}} M(f) \geq \dim_{\mathbb{C}} T(f)$$

whenever these vector-space dimensions are finite.

Remark 3.2. The finiteness of these dimensions is closely related to the condition that f defines an isolated hypersurface singularity. In the local setting at the origin, this means that the hypersurface germ

$$\{f = 0\} \subseteq (\mathbb{C}^n, 0)$$

has an isolated singularity at 0.

$$\begin{array}{ccc} \mathbb{C}[\mathbf{x}] & \twoheadrightarrow & M(f) = \mathbb{C}[\mathbf{x}]/J_f \\ & \searrow\!\!\searrow & \downarrow\!\!\downarrow \\ & & T(f) = \mathbb{C}[\mathbf{x}]/(f, J_f) \end{array}$$

The natural question is:

> When do we have
> $$\dim_{\mathbb{C}} M(f) = \dim_{\mathbb{C}} T(f)?$$

Equivalently, when does the natural surjection

$$M(f) \twoheadrightarrow T(f)$$

become an isomorphism?

Weighted Homogeneous Polynomials

Definition 3.3 (Weighted homogeneous polynomial)**.** Let

$$f = \sum_{\alpha} a_\alpha x^\alpha \in \mathbb{C}[\mathbf{x}],$$

where

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Let

$$w = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{>0}.$$

We say that f is *w-homogeneous of weighted degree d* if

$$\alpha \cdot w = \alpha_1 w_1 + \cdots + \alpha_n w_n = d$$

for every multi-index α such that $a_\alpha \neq 0$.

Remark 3.4. One may also allow rational weights

$$w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_{>0}.$$

After multiplying by a common denominator, rational weights can often be converted into integral weights.

Example 3.5. Every ordinary homogeneous polynomial of degree d is w-homogeneous with respect to

$$w = (1, \ldots, 1).$$

Example 3.6. Let

$$f = x^a + y^b.$$

Then f is weighted homogeneous with respect to

$$w = (b, a).$$

Indeed,

$$\deg_w(x^a) = ab, \qquad \deg_w(y^b) = ab.$$

Thus f has weighted degree ab.

Example 3.7. The polynomial

$$f = x^2 z - y^2$$

is weighted homogeneous with respect to

$$w = (1, 2, 2).$$

Indeed,

$$\deg_w(x^2 z) = 2 \cdot 1 + 2 = 4, \qquad \deg_w(y^2) = 2 \cdot 2 = 4.$$

Hence f has weighted degree 4.

Example 3.8. The simple surface singularities of ADE type are standard examples of weighted homogeneous isolated hypersurface singularities.

Saito's Criterion

Theorem 3.9 (Saito, 1971)**.** *Let f define an isolated hypersurface singularity. Then*

$$\dim_{\mathbb{C}} M(f) = \dim_{\mathbb{C}} T(f)$$

if and only if f is weighted homogeneous.

Remark 3.10. The equality

$$\dim_{\mathbb{C}} M(f) = \dim_{\mathbb{C}} T(f)$$

means that adding the class of f to the Jacobian ideal does not decrease the dimension. Equivalently, in the Milnor algebra,

$$f = 0.$$

Thus the equality is closely related to the condition

$$f \in J_f.$$

For weighted homogeneous polynomials this follows from the weighted Euler formula.

Weighted Euler Formula and Graded Structures

Proposition 3.11 (Weighted Euler formula)**.** *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be w-homogeneous of weighted degree d. Then*

$$df = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i}.$$

In particular,

$$f \in J_f.$$

Proof. Write

$$f = \sum_{\alpha} a_\alpha x^\alpha.$$

Since f is w-homogeneous of degree d, we have

$$\alpha \cdot w = d$$

whenever $a_\alpha \neq 0$. Now

$$\sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}(x^\alpha) = \sum_{i=1}^{n} w_i \alpha_i x^\alpha = (\alpha \cdot w) x^\alpha = d x^\alpha.$$

Multiplying by a_α and summing over all α gives

$$\sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} = df.$$

Thus

$$f = \frac{1}{d} \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} \in J_f.$$

□

Corollary 3.12. *If f is weighted homogeneous, then*

$$T(f) = \frac{\mathbb{C}[\mathbf{x}]}{(f, J_f)} = \frac{\mathbb{C}[\mathbf{x}]}{J_f} = M(f).$$

Fix weights

$$w = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{>0}.$$

Then the polynomial ring has a weighted grading

$$\mathbb{C}[\mathbf{x}] = \bigoplus_{k \geq 0} S_k,$$

where

$$S_k = \{w\text{-homogeneous polynomials of degree } k\} \cup \{0\}.$$

If f is w-homogeneous of degree d, then

$$\frac{\partial f}{\partial x_i}$$

is w-homogeneous of degree

$$d - w_i.$$

Therefore J_f is a weighted homogeneous ideal. Hence the quotient

$$T(f) = \frac{\mathbb{C}[\mathbf{x}]}{(f, J_f)}$$

inherits a weighted grading:

$$T(f) = \bigoplus_{k \geq 0} R_k,$$

where

$$R_k = \frac{S_k}{S_k \cap (f, J_f)}.$$

If f is weighted homogeneous, then $f \in J_f$, and therefore

$$R_k = \frac{S_k}{S_k \cap J_f}.$$

Derivation Lie Algebra of the Tjurina Algebra

Definition 3.13 (Derivation Lie algebra)**.** Let A be a $\mathbb{C}$-algebra. The $\mathbb{C}$-derivations of A form a Lie algebra

$$\mathrm{Der}_{\mathbb{C}}(A)$$

with bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

For the Tjurina algebra of f, we write

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f)).$$

Suppose f is w-homogeneous. Since

$$T(f) = \bigoplus_{k \geq 0} R_k$$

is graded, the derivation Lie algebra inherits a grading:

$$L(f) = \bigoplus_{j \in \mathbb{Z}} L_j,$$

where

$$L_j = \{D \in L(f) \mid D(R_k) \subseteq R_{k+j} \text{ for all } k\}.$$

Example 3.14. Let

$$\mathbb{C}[x] = \bigoplus_{k \geq 0} S_k$$

be the usual grading, where S_k is the vector space of homogeneous polynomials of degree k. Then

$$\frac{\partial}{\partial x}$$

has degree -1, because if $g \in S_k$, then

$$\frac{\partial g}{\partial x} \in S_{k-1}.$$

More generally,

$$x^j \frac{\partial}{\partial x}$$

has degree $j-1$, because

$$x^j \frac{\partial g}{\partial x} \in S_{k+j-1}.$$

Remark 3.15. The natural question is whether derivations of very negative degree can occur. For example, one asks whether an element of L_{-2} can exist.

Yau's Conjecture

Conjecture 3.16 (Yau's conjecture)**.** Let

$$f \in \mathbb{C}[x_1, \ldots, x_n]$$

be a weighted homogeneous polynomial defining an isolated hypersurface singularity. Write

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f)) = \bigoplus_{j \in \mathbb{Z}} L_j.$$

Then

$$L_j = 0 \qquad \text{for all } j < 0.$$

Known cases

The conjecture is known in several cases:

1) For $f \in \mathbb{C}[x_1, x_2]$ and $f \in \mathbb{C}[x_1, x_2, x_3]$, by work of Chen, Xu, and Yau.
2) For $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$, by work of Chen.
3) For ordinary homogeneous polynomials, by work of Xu and Yau.
4) For weights $w = (w_1, \ldots, w_n)$ satisfying $w_n \geq \frac{w_1}{2}$, assuming $w_1 \geq w_2 \geq \cdots \geq w_n$, by work of Yau and Zuo.

The Plane Curve Case

We now discuss the case

$$f \in \mathbb{C}[x, y].$$

Theorem 3.17. *Let*

$$f \in \mathbb{C}[x, y]$$

be a weighted homogeneous polynomial defining an isolated hypersurface singularity. Then the Tjurina algebra

$$T(f)$$

has no negative derivations. Equivalently,

$$L_j = 0 \qquad \textit{for all } j < 0.$$

Proof. Let

$$w = (w_1, w_2)$$

be the weight vector. Reordering the variables if necessary, assume

$$w_1 \geq w_2.$$

Since f is weighted homogeneous, the weighted Euler formula gives

$$f \in J_f.$$

Therefore

$$T(f) = \frac{\mathbb{C}[x, y]}{(f, \partial_x f, \partial_y f)} = \frac{\mathbb{C}[x, y]}{(\partial_x f, \partial_y f)}.$$

Let

$$\overline{D} : T(f) \to T(f)$$

be a homogeneous derivation of degree $-k < 0$, where $k > 0$.

The derivation $\overline{D}$ is represented by a derivation

$$D : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$$

satisfying

$$D\Big((\partial_x f, \partial_y f)\Big) \subseteq (\partial_x f, \partial_y f).$$

Write

$$D = g\frac{\partial}{\partial x} + h\frac{\partial}{\partial y},$$

where

$$g = D(x), \qquad h = D(y).$$

Because D has degree $-k$, the polynomials g and h are weighted homogeneous of degrees

$$\deg_w(g) = w_1 - k, \qquad \deg_w(h) = w_2 - k.$$

Moreover, since $\overline{D}$ is a derivation of the finite-dimensional local algebra

$$T(f),$$

it preserves the maximal ideal

$$\mathfrak{m} = (\overline{x}, \overline{y}) \subseteq T(f).$$

Thus we may take

$$g, h \in (x, y).$$

Since

$$\deg_w(h) = w_2 - k < w_2,$$

and every nonzero element of (x, y) has weighted degree at least

$$\min\{w_1, w_2\} = w_2,$$

we must have

$$h = 0.$$

We have

$$\deg_w(g) = w_1 - k < w_1.$$

Since $g \in (x, y)$ and $w_1 \geq w_2$, any monomial involving x has weighted degree at least w_1. Therefore g cannot contain x. Hence

$$g \in \mathbb{C}[y].$$

Because g is weighted homogeneous, either $g = 0$ or

$$g = cy^b$$

for some

$$c \in \mathbb{C}^\times, \qquad b \geq 1.$$

If $g = 0$, then $D = 0$, so there is nothing to prove. We therefore assume

$$g = cy^b.$$

Since

$$D\Big((\partial_x f, \partial_y f)\Big) \subseteq (\partial_x f, \partial_y f),$$

we have

$$D(\partial_x f) \in (\partial_x f, \partial_y f).$$

The weighted degree of $D(\partial_x f)$ is

$$\deg_w(D(\partial_x f)) = (d - w_1) - k = d - w_1 - k.$$

The generators $\partial_x f$ and $\partial_y f$ have weighted degrees

$$d - w_1 \qquad \text{and} \qquad d - w_2.$$

Since

$$w_1 \geq w_2,$$

we have

$$d - w_1 \leq d - w_2.$$

Thus the least weighted degree of a nonzero element of the homogeneous ideal

$$(\partial_x f, \partial_y f)$$

is at least

$$d - w_1.$$

But

$$d - w_1 - k < d - w_1.$$

Therefore

$$D(\partial_x f) = 0.$$

Now

$$D(\partial_x f) = cy^b \frac{\partial}{\partial x}(\partial_x f) = cy^b \frac{\partial^2 f}{\partial x^2}.$$

Since $\mathbb{C}[x,y]$ is a domain and $cy^b \neq 0$, we get

$$\frac{\partial^2 f}{\partial x^2} = 0.$$

Hence f is at most linear in x:

$$f = \alpha y^r + \beta x y^s$$

for some

$$\alpha, \beta \in \mathbb{C}, \qquad r \geq 2, \qquad s \geq 0.$$

Since f defines a hypersurface singularity at the origin, there are no nonzero linear terms. Hence $s \geq 1$.

Suppose

$$s \geq 2.$$

Then

$$\frac{\partial f}{\partial x} = \beta y^s,$$

and

$$\frac{\partial f}{\partial y} = \alpha r y^{r-1} + \beta s x y^{s-1}.$$

On the x-axis, namely at points with $y = 0$, both partial derivatives vanish:

$$\partial_x f(x,0) = 0, \qquad \partial_y f(x,0) = 0.$$

Thus the entire x-axis is contained in the singular locus, contradicting the assumption that f defines an isolated hypersurface singularity. Therefore

$$s = 1.$$

So

$$f = \alpha y^r + \beta x y.$$

Since f has an isolated singularity, we must have

$$\beta \neq 0.$$

Then

$$\partial_x f = \beta y.$$

Thus

$$y \in (\partial_x f, \partial_y f).$$

Since

$$g = c y^b,$$

we get

$$g \in (\partial_x f, \partial_y f).$$

Also $h = 0$. Therefore

$$D(x) = g = 0 \quad \text{in } T(f), \qquad D(y) = h = 0 \quad \text{in } T(f).$$

Hence

$$\overline{D} = 0.$$

Thus there are no nonzero negative derivations of $T(f)$. □

Remark 3.18. The essential idea is that a negative derivation lowers weighted degree. Since the maximal ideal of the Tjurina algebra starts in positive weighted degree, there is very little room for a negative derivation to act nontrivially. In two variables, this degree restriction forces the derivation to have the form

$$D = cy^b \frac{\partial}{\partial x}.$$

Then preservation of the Jacobian ideal forces

$$\partial_x^2 f = 0,$$

so f is linear in x. Finally, the isolated singularity condition forces the linear-in-x term to be exactly of the form

$$\beta xy.$$

This makes

$$y \in J_f,$$

which kills the remaining possible action of D on the Tjurina algebra.

4 Lecture 4- May 28, 2026

Let

$$S = \mathbb{C}[x_1, \ldots, x_n].$$

A polynomial $f \in S$ is *w-homogeneous*, of weighted degree d with respect to a weight vector

$$w = (w_1, \ldots, w_n)$$

if for every monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ appearing in f with a non-zero coefficient, we have

$$\sum_{i=1}^{n} w_i \alpha_i = d.$$

Definition 4.1 (Jacobian ideal, Milnor algebra, and Tjurina algebra)**.** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. The **Jacobian ideal** of f is

$$\mathrm{Jac}_f = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \subseteq S.$$

The **Milnor algebra** of f is

$$\mathcal{M}(f) = \frac{S}{\mathrm{Jac}_f}.$$

The **Tjurina algebra** of f is

$$T(f) = \frac{S}{\langle f, \mathrm{Jac}_f \rangle}.$$

Remark 4.2. If f is weighted homogeneous, then Euler's weighted identity gives

$$df = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i}.$$

Working over $\mathbb{C}$, this implies

$$\langle f, \mathrm{Jac}_f \rangle = \mathrm{Jac}_f\,.$$

Thus,

$$T(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f, \mathrm{Jac}_f \rangle} \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathrm{Jac}_f} = \mathcal{M}(f).$$

Definition 4.3 (Milnor number)**.** If $\mathcal{M}(f)$ is finite-dimensional over $\mathbb{C}$, the **Milnor number** of f is

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{M}(f).$$

Equivalently,

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle \partial_{x_1} f, \ldots, \partial_{x_n} f \rangle}.$$

First Examples

Example 4.4. Consider the following examples in two variables.

(1) Let

$$f = x^3 + y^3 \in \mathbb{C}[x, y].$$

Then

$$\mathrm{Jac}_f = \langle 3x^2, 3y^2 \rangle = \langle x^2, y^2 \rangle,$$

so

$$\mathcal{M}(f) = \frac{\mathbb{C}[x,y]}{\langle x^2, y^2\rangle}.$$

A $\mathbb{C}$-basis is

$$\{1, x, y, xy\}.$$

Hence

$$\mu(f) = 4.$$

(2) Let

$$g = x^2y + y^3.$$

Then

$$\frac{\partial g}{\partial x} = 2xy, \qquad \frac{\partial g}{\partial y} = x^2 + 3y^2,$$

so

$$\mathrm{Jac}_g = \langle xy, x^2 + 3y^2\rangle.$$

This again gives

$$\mu(g) = 4.$$

(3) Let

$$h = x^2y + xy^2.$$

Then

$$\frac{\partial h}{\partial x} = 2xy + y^2, \qquad \frac{\partial h}{\partial y} = x^2 + 2xy.$$

This gives

$$\mu(h) = 4.$$

Notice that f, g, and h are homogeneous of degree 3 with respect to the standard weight vector

$$w = (1, 1).$$

The Milnor–Orlik Formula

Theorem 4.5 (Milnor–Orlik, 1970)**.** *Let*

$$f \in \mathbb{C}[x_1, \dots, x_n]$$

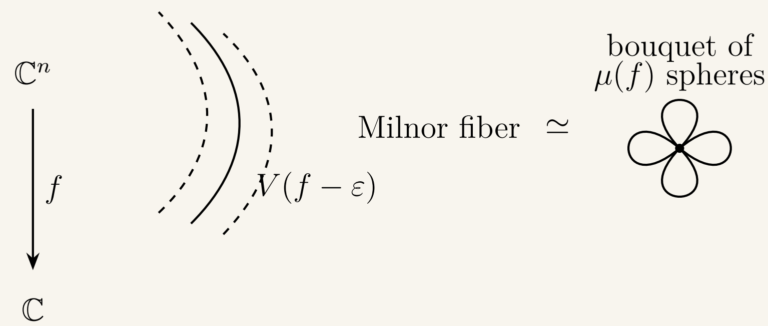
be a weighted homogeneous polynomial of degree d with respect to the weights

$$w = (w_1, \dots, w_n).$$

Then

$$\mu(f) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

Remark 4.6. The original proof is topological. If f has an isolated hypersurface singularity at the origin, then the Milnor fiber has the homotopy type of a bouquet of $\mu(f)$ spheres of real dimension $n-1$.



Algebraic Verification for

Example 4.7. Let

$$f = x^3 + y^3, \qquad S = \mathbb{C}[x,y].$$

Here

$$w = (1,1).$$

Using the formula,

$$\mu(f) = \left(\frac{3-1}{1}\right)\left(\frac{3-1}{1}\right) = 4.$$

We now re-verify this algebraically. Since

$$\mathrm{Jac}_f = \langle x^2, y^2 \rangle,$$

we have

$$0 \longrightarrow \langle x^2, y^2 \rangle \longrightarrow S \longrightarrow \mathcal{M}(f) \longrightarrow 0.$$

Consider the map

$$\varphi : S \oplus S \longrightarrow \langle x^2, y^2 \rangle$$

given by

$$\varphi(g,h) = gx^2 + hy^2.$$

$$\begin{array}{c} S \oplus S \\ \downarrow \varphi \\ \langle x^2, y^2 \rangle \hookrightarrow S \longrightarrow \mathcal{M}(f) \longrightarrow 0. \end{array}$$

The kernel is generated by

$$Q = S(y^2, -x^2),$$

since

$$y^2x^2 - x^2y^2 = 0.$$

Putting this together gives an exact sequence

$$0 \longrightarrow S \xrightarrow{\varphi_2} S \oplus S \xrightarrow{\varphi_1} S \longrightarrow \mathcal{M}(f) \longrightarrow 0,$$

where

$$\varphi_2(1) = (y^2, -x^2), \qquad \varphi_1(g, h) = gx^2 + hy^2.$$

$$0 \longrightarrow S \xrightarrow{\varphi_2} S \oplus S \xrightarrow{\varphi_1} S \longrightarrow \mathcal{M}(f) \longrightarrow 0$$

$$1 \longmapsto (y^2, -x^2) \qquad \in \qquad (g, h) \longmapsto gx^2 + hy^2 \longmapsto .$$

However, this sequence does not yet respect the natural grading. For example, if $K \in S$ is homogeneous of degree i, then

$$K \longmapsto K(y^2, -x^2)$$

which implies degree-zero maps with respect to the standard grading.
To fix this, we shift the grading. For a graded module M, the shifted module $M(-d)$ is defined by

$$M(-d)_i = M_{i-d}.$$

We then have

$$0 \longrightarrow S(-4) \xrightarrow{\varphi_2} S(-2) \oplus S(-2) \xrightarrow{\varphi_1} S \longrightarrow \mathcal{M}(f) \longrightarrow 0.$$

Hilbert Series Computation

Definition 4.8 (Hilbert series)**.** Let

$$R = \bigoplus_{i \geq 0} R_i$$

be a graded ring with finite-dimensional graded pieces. The **Hilbert series** of R is

$$\mathrm{HS}_R(t) = \sum_{i \geq 0} \dim_{\mathbb{C}}(R_i) t^i.$$

Proposition 4.9 (Additivity of Hilbert series)**.** *Given a short exact sequence of graded $\mathbb{C}$-vector spaces*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we have

$$\mathrm{HS}_B(t) = \mathrm{HS}_A(t) + \mathrm{HS}_C(t).$$

Equivalently,

$$\mathrm{HS}_C(t) = \mathrm{HS}_B(t) - \mathrm{HS}_A(t).$$

Example 4.10. Apply additivity to the Koszul resolution

$$0 \longrightarrow S(-4) \longrightarrow S(-2) \oplus S(-2) \longrightarrow S \longrightarrow \mathcal{M}(f) \longrightarrow 0.$$

We get

$$\mathrm{HS}_{\mathcal{M}(f)}(t) = \mathrm{HS}_S(t) - \mathrm{HS}_{S(-2)\oplus S(-2)}(t) + \mathrm{HS}_{S(-4)}(t).$$

Since

$$S = \mathbb{C}[x, y],$$

we have

$$\mathrm{HS}_S(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + \cdots = \frac{1}{(1-t)^2}.$$

$$\mathrm{HS}_{S(-2)\oplus S(-2)}(t) = 2t^2\,\mathrm{HS}_S(t) = \frac{2t^2}{(1-t)^2},$$

$$\mathrm{HS}_{S(-4)}(t) = t^4\,\mathrm{HS}_S(t) = \frac{t^4}{(1-t)^2}.$$

$$\mathrm{HS}_{\mathcal{M}(f)}(t) = \frac{1}{(1-t)^2} - \frac{2t^2}{(1-t)^2} + \frac{t^4}{(1-t)^2}$$

$$= \frac{1 - 2t^2 + t^4}{(1-t)^2}$$
$$= \frac{(1-t^2)^2}{(1-t)^2}$$
$$= (1+t)^2$$
$$= 1 + 2t + t^2.$$

Thus

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{M}(f) = \mathrm{HS}_{\mathcal{M}(f)}(1) = 1 + 2 + 1 = 4.$$

Algebraic Proof of the Milnor–Orlik Formula

Theorem 4.11 (Algebraic proof of Milnor–Orlik, after Y. Soler)**.** *Let*

$$f \in S = \mathbb{C}[x_1, \dots, x_n]$$

be a weighted homogeneous polynomial of degree d with respect to

$$w = (w_1, \dots, w_n).$$

Then

$$\mu(f) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

Proof. Since f defines an isolated hypersurface singularity, the Jacobian ideal

$$\mathrm{Jac}_f = \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle$$

is generated by a regular sequence

$$\partial_{x_1} f, \dots, \partial_{x_n} f$$

Because f is weighted homogeneous of degree d, each partial derivative $\partial_{x_i} f$ is weighted homogeneous of degree

$$d - w_i.$$

The Koszul complex on

$$\partial_{x_1} f, \dots, \partial_{x_n} f$$

gives a free resolution

$$0 \longrightarrow S\left(-\sum_{i=1}^{n}(d-w_i)\right) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{n} S(-(d-w_i)) \longrightarrow S \longrightarrow \mathcal{M}(f) \longrightarrow 0.$$

The top shift is

$$\sum_{i=1}^{n}(d-w_i) = nd - \sum_{i=1}^{n} w_i,$$

so the free module is

$$S(-(nd - w_1 - \cdots - w_n)).$$

Using additivity of Hilbert series on this Koszul resolution, we obtain

$$\mathrm{HS}_{\mathcal{M}(f)}(t) = \mathrm{HS}_S(t) \prod_{i=1}^{n} \left(1 - t^{d-w_i}\right).$$

We know

$$\mathrm{HS}_S(t) = \frac{1}{\prod_{i=1}^{n}(1-t^{w_i})}.$$

Hence

$$\mathrm{HS}_{\mathcal{M}(f)}(t) = \prod_{i=1}^{n} \frac{1-t^{d-w_i}}{1-t^{w_i}}.$$

Evaluating at $t = 1$ gives the total vector-space dimension:

$$\mu(f) = \mathrm{HS}_{\mathcal{M}(f)}(1) = \lim_{t\to 1} \mathrm{HS}_{\mathcal{M}(f)}(t).$$

Therefore

$$\mu(f) = \lim_{t\to 1} \prod_{i=1}^{n} \frac{1-t^{d-w_i}}{1-t^{w_i}} = \prod_{i=1}^{n} \lim_{t\to 1} \frac{1-t^{d-w_i}}{1-t^{w_i}}.$$

By L'Hôpital's Rule or factorization,

$$\lim_{t\to 1} \frac{1-t^{d-w_i}}{1-t^{w_i}} = \frac{d-w_i}{w_i}.$$

Hence

$$\mu(f) = \prod_{i=1}^{n} \frac{d-w_i}{w_i}.$$

□

The Yau Algebra

Definition and First Examples

Definition 4.12 (Yau algebra)**.** Let $f \in \mathbb{C}[x_1, \dots, x_n]$ define an isolated hypersurface singularity. The **Yau algebra** of f is the Lie algebra

$$L(f) = \mathrm{Der}_{\mathbb{C}}(T(f)).$$

If f is weighted homogeneous, then

$$L(f) \cong \mathrm{Der}_{\mathbb{C}}(\mathcal{M}(f)).$$

Example 4.13. For weighted homogeneous plane curve singularities, one may compare the dimension of the Yau algebra for different defining equations.

(1) If

$$f = x^4 + y^4,$$

then

$$\dim_{\mathbb{C}} L(f) = 12.$$

(2) If

$$g = x^3 y + y^4,$$

then

$$\dim_{\mathbb{C}} L(g) = 11.$$

Yau–Zuo's Conjecture

Conjecture 4.14 (Yau–Zuo)**.** Let

$$F = x_1^{a_1} + \cdots + x_n^{a_n}$$

be a Brieskorn-Pham polynomial with weights

$$w = \left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right).$$

Let g be another polynomial with respect to the same weights w. Assume that g defines an isolated hypersurface singularity. Then

$$\dim_{\mathbb{C}} L(g) \leq \dim_{\mathbb{C}} L(F).$$

Computing the Yau Algebra of the Brieskorn–Pham Polynomial

Let

$$F = x_1^{a_1} + \cdots + x_n^{a_n}.$$

The Jacobian ideal is

$$\mathrm{Jac}_F = \left\langle x_1^{a_1-1}, \ldots, x_n^{a_n-1} \right\rangle,$$

Since F is weighted homogeneous, the Tjurina and Milnor algebras agree:

$$T(F) \cong \mathcal{M}(F) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle x_1^{a_1-1}, \ldots, x_n^{a_n-1} \rangle}.$$

This factors as a tensor product:

$$\frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle x_1^{a_1-1}, \ldots, x_n^{a_n-1} \rangle} \cong \frac{\mathbb{C}[x_1]}{\langle x_1^{a_1-1} \rangle} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \frac{\mathbb{C}[x_n]}{\langle x_n^{a_n-1} \rangle}.$$

Remark 4.15 (Added Context)**.** This tensor product decomposition is useful because derivations of tensor products over a field split according to the factors. Namely, if A and B are $\mathbb{C}$-algebras, then

$$\mathrm{Der}_{\mathbb{C}}(A \otimes_{\mathbb{C}} B) \cong \mathrm{Der}_{\mathbb{C}}(A) \otimes_{\mathbb{C}} B \oplus A \otimes_{\mathbb{C}} \mathrm{Der}_{\mathbb{C}}(B),$$

by acting trivially on the other factor.

Thus, recalling the derivation property for tensor products,

$$\mathrm{Der}_{\mathbb{C}}(A \otimes_{\mathbb{C}} B) = \mathrm{Der}_{\mathbb{C}}(A) \otimes_{\mathbb{C}} B + A \otimes_{\mathbb{C}} \mathrm{Der}_{\mathbb{C}}(B),$$

Set

$$A_i = \frac{\mathbb{C}[x_i]}{\langle x_i^{a_i-1} \rangle}.$$

Notice that

$$\dim_{\mathbb{C}} A_i = a_i - 1.$$

Then

$$\mu(F) = \dim_{\mathbb{C}} \mathcal{M}(F) = \prod_{i=1}^{n} (a_i - 1).$$

For the one-variable algebra

$$A_i = \frac{\mathbb{C}[x_i]}{\langle x_i^{a_i-1} \rangle},$$

the condition that a derivation preserve the relation $x_i^{a_i-1} = 0$ forces the image to lie in the ideal generated by x_i. Hence

$$\dim_{\mathbb{C}} \operatorname{Der}_{\mathbb{C}}(A_i) = a_i - 2.$$

$$\begin{aligned}\dim_{\mathbb{C}} L(F) &= \sum_{i=1}^{n} \dim_{\mathbb{C}} \operatorname{Der}_{\mathbb{C}}(A_i) \prod_{\substack{1\le j\le n\\ j\ne i}} \dim_{\mathbb{C}} A_j\\ &= \sum_{i=1}^{n}(a_i-2) \prod_{\substack{1\le j\le n\\ j\ne i}} (a_j-1).\end{aligned}$$

Equivalently,

$$\dim_{\mathbb{C}} L(F) = n\mu(F) - \sum_{i=1}^{n} \prod_{\substack{1\le j\le n\\ j\ne i}} (a_j-1).$$

Or written differently,

$$\dim_{\mathbb{C}} L(F) = n\mu(F) - \sum_{i=1}^{n} (a_1-1)\cdots\widehat{(a_i-1)}\cdots(a_n-1),$$

5 Lecture 5- May 29, 2026

Some higher-order versions of $\mathcal{T}(f) = \frac{\mathbb{C}[\mathbf{x}]}{\langle f, J_f\rangle}$:

1. Dimca, Sticlaru (2014): $\langle f, \text{minors of } \operatorname{Hess}(f)\rangle$
2. Greuel, Pham (2017): $\langle f, \mathfrak{m}^k J_f\rangle$
3. Hussain, Yau, Zuo (2024): $\langle f, J_f, J_f^2, \dots, J_f^k\rangle$

Higher-order Jacobian matrix

Definition 5.1. Let $f \in \mathbb{C}[x_1, \dots, x_s]$, and $n \ge 1$. We define the higher-order Jacobian matrix as:

$$Jac_n(f) = \left(\frac{1}{(\alpha-\beta)!} \frac{\partial^{\alpha-\beta}(f)}{\partial x^{\alpha-\beta}} \right)_{\substack{\alpha\in\mathbb{N}^s,\ 1\le|\alpha|\le n\\ \beta\in\mathbb{N}^s,\ 0\le|\beta|\le n-1}}$$

Remark 5.2. The (β, α) entry is 0 if $\alpha_i < \beta_i$.

Examples:

For $n = 1$,

$$Jac_1(f) = (\partial_{x_1} f, \ldots, \partial_{x_s} f)$$

For $n = 2$, $s = 2$,

$$Jac_2(f) = \begin{matrix} \\ (0,0) \\ (1,0) \\ (0,1) \end{matrix} \quad \begin{matrix} (1,0) & (0,1) & (2,0) & (1,1) & (0,2) \\ \end{matrix} \left[\begin{matrix} \partial_{x_1} f & \partial_{x_2} f & \frac{1}{2!}\partial^2_{x_1} f & \partial_{x_1 x_2} f & \frac{1}{2!}\partial^2_{x_2} f \\ f & 0 & \partial_{x_1} f & \partial_{x_2} f & 0 \\ 0 & f & 0 & \partial_{x_1} f & \partial_{x_2} f \end{matrix} \right]$$

Kähler Differentials.
Let R be a k-algebra. A derivation $d : R \to M$ satisfies the Leibniz rule, where M is an R-module.

$$I = \ker \begin{pmatrix} R \otimes_k R \to R \\ a \otimes b \mapsto ab \end{pmatrix}$$

$$\Omega^1_{R/k} = I/I^2$$

Then, $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_{R\text{-mod}}(\Omega^1_{R/k}, M)$.

Furthermore, we define the higher order modules:

$$I/I^{n+1} = \mathcal{J}^n_{R/k}$$

For $R = \frac{k[x_1,\ldots,x_s]}{\langle f_1,\ldots,f_r \rangle}$, we have the exact sequence:

$$R^r \xrightarrow{Jac_n(f_1,\ldots,f_r)^t} R^S \to \mathcal{J}^n_{R/k} \to 0$$

Hussain, Ma, Yau, Zuo (2023):

Definition 5.3. Let $\mathcal{J}_n(f) = \langle \text{maximal minors of } Jac_n(f) \rangle$. We define the higher-order Tjurina algebras as:

$$\mathcal{T}_n(f) = \frac{\mathbb{C}[x]}{\langle f, \mathcal{J}_n(f) \rangle} \quad \text{or} \quad \mathcal{T}_n(f) = \frac{\mathbb{C}\{x\}}{\langle f, \mathcal{J}_n(f) \rangle}$$

and the derivation module:

$$L_n(f) = \mathrm{Der}_{\mathbb{C}}(\mathcal{T}_n(f))$$

Conjecture 5.4. There are higher-order versions of Day 1 – Day 4 results.

Day 1: Higher order Mather-Yau Theorem

Theorem 5.5. *Let $f, g \in \mathbb{C}\{x\}$ define an Isolated Hypersurface Singularity (I.H.S.). Then,*

$$V(f) \cong V(g) \iff \mathcal{T}_n(f) \cong \mathcal{T}_n(g) \text{ for some } n.$$

($\implies$) [Le, Yasuda, 2024]: Proved using Fitting Ideals.

($\impliedby$) [Nguyen, 2026]

Day 2: Higher-order classification of ADE singularities

Theorem 5.6 (Fan, Hussain, Yau, Zuo, 2025)**.** *Let X, Y be ADE singularities. Then*

$$X \cong Y \iff L_2(X) \cong L_2(Y).$$

Dream: Given any $X \ncong Y$, I.H.S., is $L_n(X) \ncong L_n(Y)$ for some $n \gg 0$?

- $n = 1$: Differentiate using $\dim(L(\cdot))$, $\operatorname{rk}(L(\cdot))$.
- $n = 2$: Just the dimension differentiates them.

Day 3: Higher order version of Yau's conjecture

Notice:

f is w-homogeneous $\implies$ $\partial_{x_i} f$ is w-homogeneous
$\implies \langle J_f \rangle$ is w-homogeneous.
$\implies f$ is w-homogeneous $\implies$ minors of $Jac_n(f)$ are w-homogeneous. [Bedilla, Castano, Daniel]
$\implies \langle f, \mathcal{J}_n(f) \rangle$ is w-homogeneous.

Theorem 5.7 (B, C, D, Nunez Betancourt, 2025)**.** *There are no negative w-degree derivations on $\mathcal{T}_n(f)$ for $n \geq 2$.*

Remark 5.8. Note that $f \notin \mathcal{J}_n(f)$ for $n \geq 2$.

Key Ideas:

- **$n = 1$:** The proof in all known cases starts with setting $w = \deg(\partial_{x_1} f)$ (assuming $w_1 \geq w_2 \geq \cdots \geq w_s$). This implies $D(\partial_{x_1} f) = 0$.
- **$n \geq 2$:**

- *Step 1:* $w\text{-}\deg(f) \leq w\text{-}\deg(\text{max minor})$
- *Step 2:* $D(f) = 0 \implies \ldots$

Day 4: Higher Order version of Milnor-Orlik

Example:

$$f = x^4 + y^4 + z^4 \implies \dim_{\mathbb{C}} \mathcal{T}_2(f) = 242$$
$$g = x^4 + y^4 + yz^3 \implies \dim_{\mathbb{C}} \mathcal{T}_2(g) = 239$$

Conjecture 5.9. Let $f = x_1^{a_1} + \cdots + x_s^{a_s}$ and $w = \left(\frac{1}{a_1}, \ldots, \frac{1}{a_s}\right)$. Let g be w-homogeneous of degree 1. Then,

$$\dim_{\mathbb{C}} \mathcal{T}_n(g) \leq \dim_{\mathbb{C}} \mathcal{T}_n(f).$$

Higher-order version of Yau-Zuo's conjecture:

$$\boxed{\dim_{\mathbb{C}} L_n(g) \leq \dim_{\mathbb{C}} L_n(f)}$$

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