PASCA 2.0

Summer School at CIMAT

CIMAT

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Lecture Notes

Singularities of Foliations in Characteristic p

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1 An Invitation to p-Singularities

We begin with some motivating questions about foliations, differential equations, and singularities in characteristic p.

Question 1.1. What does it mean for a PDE, or more geometrically for a singular space governed by differential equations, to be F-rational?

Question 1.2. What is a foliation?

Geometric Motivation

Example 1.3 (Elementary pictures of foliations)**.** Consider a small open interval or disk, for instance

$$X = (-\epsilon, \epsilon)$$

(1) The equation

$$x^2 + y^2 = c$$

(2) The equation

$$\frac{y}{x} = c$$

These two pictures represent two basic ways in which a space may be decomposed into lower-dimensional leaves.

Remark 1.4 (Added Context)**.** A foliation is a decomposition of a space into subspaces called leaves, which locally look like parallel slices. In differential geometry, these leaves are usually integral manifolds of a distribution, i.e. of a subbundle of the tangent bundle.

Foliations on Manifolds

Let X be a smooth manifold and let T_X denote its tangent bundle. A distribution is a subbundle

$$F \subseteq T_X.$$

Theorem 1.5 (Frobenius Theorem)**.** *Let X be a smooth manifold and let F be a subbundle. Then F is integrable if and only if the following conditions hold:*

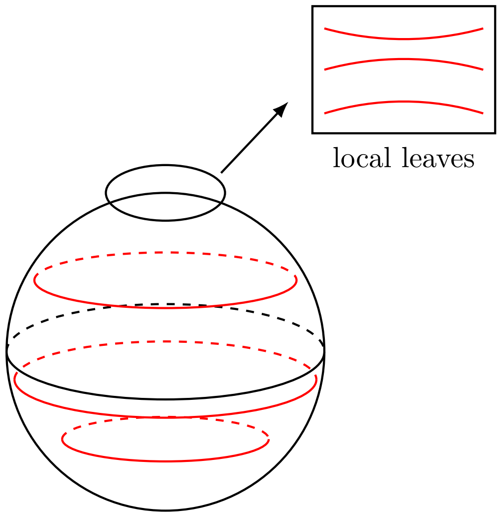

Figure 1. A schematic foliation by leaves on a surface, with a local chart.

(1) F is closed under the Lie bracket of vector fields, i.e.

$$[F, F] \subseteq F;$$

Thus one has the following dictionary:

$$\left\{ \text{ foliations } \mathcal{F} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{integrable subbundles} \\ T_{\mathcal{F}} = D \subseteq T_X \end{array} \right\}.$$

Example 1.6 (Concentric-circle foliation)**.** The foliation defined by the level sets

$$x^2 + y^2 = c$$

yields:

$$d(x^2 + y^2) = 0.$$

$$2x\,dx + 2y\,dy = 0,$$

$$x\,dx + y\,dy = 0.$$

$$\mathcal{F} = \int D.$$

Foliations in Commutative Algebra

We now translate the differential-geometric notion into commutative algebra.

Let k be a field and let A be a k-algebra which is a normal integral domain. Let

$$\mathrm{Der}_k(A)$$

denote the module of derivations.

Definition 1.7 (Algebraic foliation)**.** A *foliation* on A is an A-submodule

$$\mathcal{F} \subseteq \mathrm{Der}_k(A)$$

satisfying:

(1) **Saturation:** the quotient

$$\mathrm{Der}_k(A)/\mathcal{F}$$

is torsion-free.

(2) **Lie-closedness:** $\mathcal{F}$ is closed under the Lie bracket:

$$[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}.$$

Remark 1.8. The saturation condition is the algebraic analogue of requiring the quotient to behave like a vector bundle on a dense open set. Since A is normal, torsion-freeness is a natural replacement for local freeness in singular settings.

The Characteristic p Miracle

Assume now that k has characteristic $p > 0$. In characteristic p, foliations are closely related to intermediate rings between A^p and A.

1.1 The Ekedahl–Jacobson correspondence

Theorem 1.9 (Ekedahl–Jacobson Correspondence)**.** *Let A be a normal domain over a field k of characteristic p. There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} R \textit{ normal such that} \\ A^p \subseteq R \subseteq A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{F} \subseteq \mathrm{Der}_k(A) \\ \mathcal{F} \textit{ a foliation} \end{array} \right\}.$$

given by:

$$R \longmapsto \mathrm{Der}_R(A)$$

$$\mathcal{F} \longmapsto A^{\mathcal{F}},$$

where

$$A^{\mathcal{F}} = \{a \in A \mid \partial(a) = 0 \text{ for all } \partial \in \mathcal{F}\}.$$

Remark 1.10. The containment $A^p \subseteq A^{\mathcal{F}}$ holds because every derivation kills p-th powers in characteristic p:

$$\partial(a^p) = pa^{p-1}\partial(a) = 0.$$

$$\left\{ \begin{array}{c} A^p \subseteq R \subseteq A \\ R \text{ normal} \end{array} \right\} \underset{\mathcal{F} \mapsto A^{\mathcal{F}}}{\overset{R \mapsto \mathrm{Der}_R(A)}{\rightleftarrows}} \left\{ \begin{array}{c} \mathcal{F} \subseteq \mathrm{Der}_k(A) \\ \mathcal{F} \text{ foliation} \end{array} \right\}$$

A foliated Kunz theorem

Theorem 1.11 (Foliated Kunz Theorem)**.** *Let A be a smooth k-algebra of characteristic p. Under the Ekedahl–Jacobson correspondence*

$$A^{\mathcal{F}} \longleftrightarrow \mathcal{F},$$

the following are equivalent:

(1) $A^{\mathcal{F}}$, equivalently A, is smooth;

(2) A is flat over $A^{\mathcal{F}}$;

(3) A is locally free, equivalently flat, over $A^{\mathcal{F}}$.

Remark 1.12. This statement is a foliated analogue of Kunz's theorem. Classical Kunz says that a Noetherian ring of characteristic p is regular if and only if the Frobenius morphism is flat. Here, flatness of the quotient by the foliation detects smoothness of the ring of constants.

Examples

Example 1.13 (Euler derivation on the affine plane)**.** Let

$$A = k[x, y]$$

and consider the Euler derivation:

$$\partial = x\partial_x + y\partial_y.$$

We define the foliation:

$$\mathcal{F} = \langle x\partial_x + y\partial_y \rangle_A.$$

Notice that:

$$\partial(x^a y^b) = (a+b)x^a y^b.$$

This is zero if and only if:

$$a + b \equiv 0 \pmod p.$$

Thus, the constants include:

$$x^p, \quad x^{p-1}y, \quad x^{p-2}y^2, \quad \dots, \quad xy^{p-1}, \quad y^p$$

giving:

$$A^{\mathcal{F}} = k[x^p, x^{p-1}y, x^{p-2}y^2, \dots, xy^{p-1}, y^p].$$

Remark 1.14. More precisely, the invariant ring is generated by all monomials $x^a y^b$ with $a + b \equiv 0 \pmod p$. The displayed generators give the degree-p Veronese subring; depending on the precise foliation and saturation convention, one should verify whether the full constant ring is exactly this Veronese subring or its integral closure inside A. In the standard Euler-derivation example, the p-th Veronese description captures the intended lecture-note picture.

F-Rationality and Differential Operators

We now recall the general philosophy behind p-singularities.

Let A be a local ring of characteristic p. Roughly speaking, A is F-rational if it is Cohen–Macaulay and parameter ideals are tightly closed. Equivalently, for excellent local rings, F-rationality can be detected by the behavior of Frobenius actions on local cohomology or by trace maps on canonical modules.

Remark 1.15. We have a connection between foliations, differential operators, and F-rationality. The exact formulation depends on the chosen singularity class: F-rational, F-regular, or F-pure. The common theme is that Frobenius-linear maps and Cartier operators control the singularities.

Suppose that A is finite over A^p. For instance, this occurs when A is F-finite. Let ω_A be a canonical module. In the regular or finite setting one may write

$$\omega_A \cong \mathrm{Hom}_{A^p}(A, A^p),$$

For F-singularities, one studies maps induced by Frobenius trace:

$$F_*^e \omega_A \longrightarrow \omega_A.$$

Or equivalently:

$$F_*^e \omega_A \xrightarrow{F_*^e c} F_*^e \omega_A \longrightarrow \omega_A.$$

$$F_*^e \omega_A \xrightarrow{F \cdot c} F_*^e \omega_A \longrightarrow \frac{\omega_A}{(\mathrm{Sp})}.$$

Differential operators in characteristic p

The ring of k-linear differential operators on A can be described by

$$D_A = \bigcup_{e \geq 0} \mathrm{Hom}_{A^{p^e}}(A, A) \subseteq \mathrm{Hom}_k(A, A).$$

We denote the level-e differential operators as:

$$D_A^{(e)} := \mathrm{Hom}_{A^{p^e}}(A, A)$$

A foliation $\mathcal{F}$ may be viewed as an A-submodule, or more generally as part of a filtered structure inside the ring of differential operators. We denote its level-e component by

$$\mathcal{F}^{(e)} = \mathcal{F} \cap D_A^{(e)}.$$

The corresponding rings of constants form a descending chain

$$A \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots \supseteq A^{(e)} \supseteq \cdots,$$

where:

$$A^{(e)} = A^{\mathcal{F}^{(e)}}.$$

Explicitly:

$$A^{(e)} = \{a \in A \mid [\delta, a] = 0 \text{ for all } \delta \in \mathcal{F}^{(e)}\}.$$

Remark 1.16. The commutator notation $[\delta, a]$ views a as the operator "multiplication by a." Thus $[\delta, a] = 0$ means that the operator δ commutes with multiplication by a. This is a natural higher-level analogue of saying that a derivation kills a.

Divided Powers and Higher Constants

Example 1.17 (Higher constants for the Euler foliation)**.** Let

$$A = k[x, y]$$

and:

$$\partial = x\partial_x + y\partial_y.$$

We can form the module of divided powers:

$$M = \bigoplus_{k\geq 0} A\,\partial^{(k)},$$

where:

$$\partial^{(k)} = \sum_{i+j=k} x^i y^j \partial_x^{(i)} \partial_y^{(j)}.$$

The e-th constant ring is described by

$$A^{(e)} = k[x^{p^e}, x^{p^e-1}y, x^{p^e-2}y^2, \dots, xy^{p^e-1}, y^{p^e}].$$

With:

$$K_0 := \mathrm{Der}(A^{(e)})$$

Remark 1.18. The appearance of the generators

$$x^{p^e}, x^{p^e-1}y, \dots, y^{p^e}$$

reflects the higher-order constants under the divided power derivations.

Cartier Foliations

Definition 1.19 (F-foliation or Cartier foliation)**.** An *F-foliation*, also called a *Cartier foliation*, is a compatible higher-level foliation for which the corresponding system of constant rings behaves stably in the sense that

$$A^{(e)} = A^{(e+1)} \quad \text{for } e \gg 0,$$

Remark 1.20. In many treatments, a Cartier structure is not literally defined by equality $A^{(e)} = A^{(e+1)}$ for large e, but rather by compatible Frobenius-linear or Cartier-linear maps at all levels. The stabilization condition above records the intended lecture-note heuristic: an F-foliation is a foliation equipped with coherent data through all Frobenius levels.

In the local setting, let A be a regular local ring. One studies $\mathcal{F}$ as an F-regular Cartier foliation of rank r. Such a foliation is locally governed by variables

$$x_1, \dots, x_r$$

Definition 1.21 (Partial p-basis)**.** Let A be an F-finite ring of characteristic p. A collection

$$x_1, \dots, x_r \in A$$

forms a partial p-basis if they extend to a p-basis of A over A^p.

Theorem 1.22 (Cartier operators attached to a Cartier foliation)**.** *For a Cartier foliation $\mathcal{F}$, there is a compatible system of Cartier operators*

$$\kappa_e : F^e_* \omega_{\mathcal{F}_e} \longrightarrow \omega_{\mathcal{F}_e},$$

Remark 1.23. The operators κ_e are the foliated analogue of Frobenius trace maps. Properties such as F-purity, sharp F-purity, F-regularity, and F-rationality are often formulated in terms of the surjectivity, splitting, or generation behavior of such Cartier operators.

Summary of the Main Dictionary

The lecture develops the following chain of ideas:

$$\text{foliations} \quad \longleftrightarrow \quad \text{Lie-closed saturated submodules of } \operatorname{Der}_k(A)$$

$$\text{foliations} \quad \longleftrightarrow \quad \text{intermediate rings } A^p \subseteq R \subseteq A.$$

Using the ring of differential operators:

$$D_A = \bigcup_{e \geq 0} \operatorname{Hom}_{A^{p^e}}(A, A),$$

we get the following coherent picture:

$$\begin{array}{ccc} \text{foliation } \mathcal{F} & \xrightarrow{\text{constants}} & A^{\mathcal{F}} \\ {\scriptstyle \text{higher levels}}\downarrow & & \downarrow{\scriptstyle p^e\text{-level constants}} \\ \mathcal{F}^{(e)} \subseteq D_A^{(e)} & \xrightarrow{\text{constants}} & A^{(e)} \end{array}$$

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