PASCA 2.0

Summer School at CIMAT

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Lecture Notes

A Cluster of Algebra, Geometry, and Combinatorics

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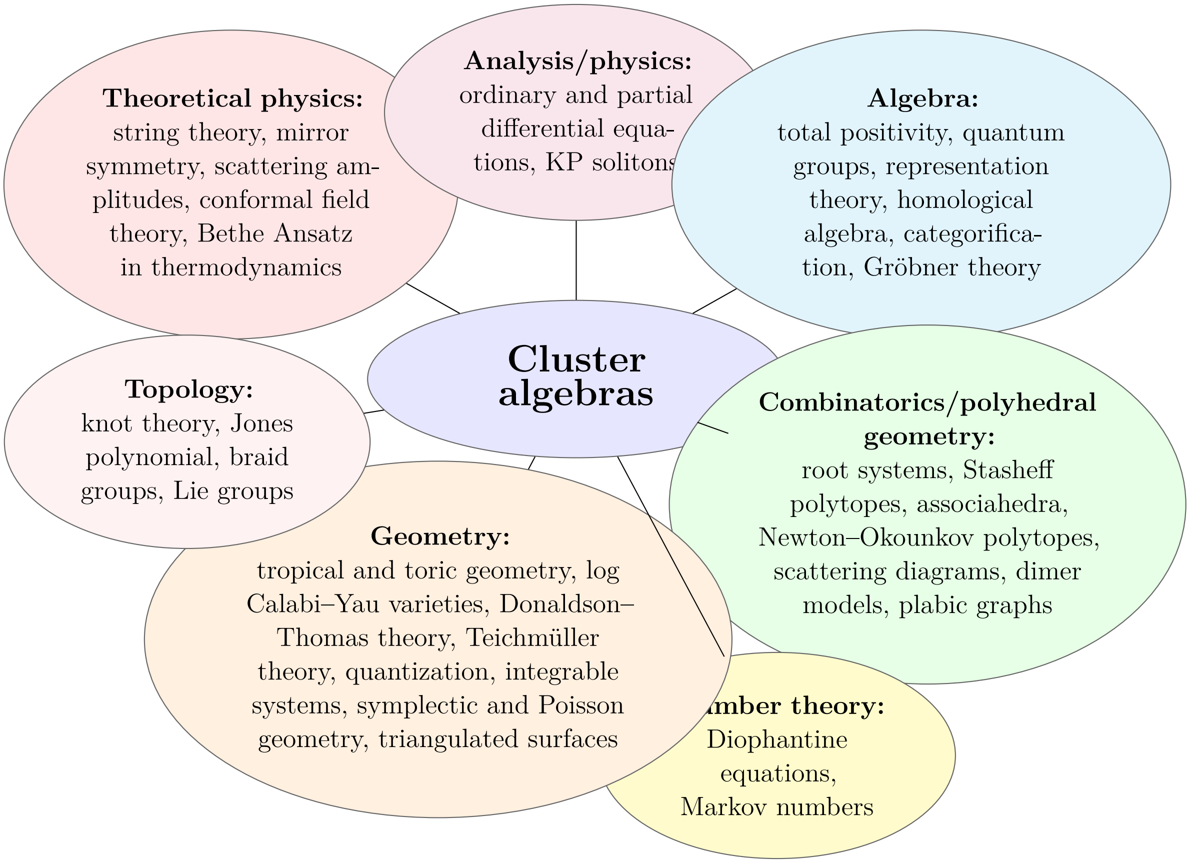
Contents

1 Lecture 1: Cluster Algebras

1.1 Cluster algebras as a meeting point

Cluster algebras arise naturally in several areas of mathematics and mathematical physics. The following diagram records the main themes emphasized in the lecture.



1.2 Historical background

- Sergei Fomin and Andrei Zelevinsky introduced cluster algebras in 2001.
- Their original motivation came from structures appearing in total positivity, quantum groups, and canonical bases.
- Cluster algebras now have their own Mathematics Subject Classification code: **13F60**.
- As of 2018, there were more than 1,755 articles and, since 2003, more than 139 events devoted to cluster algebras.

Remark 1.1. The central idea is to construct a commutative algebra from overlapping finite generating sets, called *clusters*, related by explicit birational transformations called *mutations*. The fundamental surprise is that repeated mutation often produces highly structured algebraic, combinatorial, and geometric objects.

1.3 Total Positivity and the Origin of Cluster Algebras

1.3.1 Totally positive 2×4 matrices

Let

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \in \mathbb{R}^{2\times 4}$$

Definition 1.2 (Total positivity)**.** The matrix M is called *totally positive* if all its 2×2 minors are positive:

$$\Delta_{ij}(M) := \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = a_i b_j - a_j b_i > 0 \qquad \text{for all } 1 \le i < j \le 4.$$

Example 1.3. Let

$$M = \begin{pmatrix} 1 & 0 & -3 & -5 \\ 0 & 2 & 1 & 1 \end{pmatrix}.$$

Then

$$\Delta_{12} = 2, \qquad \Delta_{13} = 1, \qquad \Delta_{14} = 1, \qquad \Delta_{23} = 6, \qquad \Delta_{24} = 10, \qquad \Delta_{34} = 2.$$

Exercise 1.4 (The Plücker relation for $\mathrm{Gr}(2,4)$)**.** Every $M \in \mathbb{R}^{2\times 4}$ satisfies

$$\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23} = \Delta_{13}\Delta_{24}.$$

Proof. Write the four columns of M as v_1, v_2, v_3, v_4, and let

$$\Delta_{ij} = \det(v_i, v_j).$$

Then

$$\det(v_1, v_2)\det(v_3, v_4) - \det(v_1, v_3)\det(v_2, v_4) + \det(v_1, v_4)\det(v_2, v_3) = 0.$$

This means

$$\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23} = \Delta_{13}\Delta_{24}.$$

For the matrix M in the previous example, this identity reads

$$2 \cdot 2 + 1 \cdot 6 = 1 \cdot 10.$$

$\square$

Question 1.5. How can one efficiently verify whether M is totally positive?

Proposition 1.6 (A positivity test for 2×4 matrices)**.** *Let M have rank 2. Then M is totally positive if and only if*

$$\Delta_{12},\ \Delta_{13},\ \Delta_{14},\ \Delta_{23},\ \Delta_{34} > 0.$$

Proof. If M is totally positive, then all six minors are positive, so the displayed five minors are positive.

Conversely, suppose

$$\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{34} > 0.$$

Then

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23},$$

so

$$\Delta_{24} = \frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}}{\Delta_{13}}.$$

This must be strictly positive since the numerator and denominator are strictly positive. $\square$

Definition 1.7 (Positivity test)**.** A subset of Plücker coordinates is called a *positivity test* if positivity of those coordinates implies positivity of all Plücker coordinates.

In the case above,

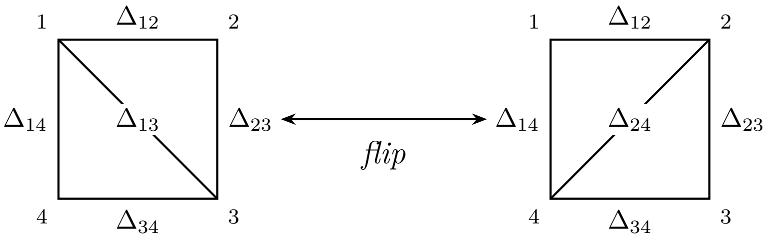
$$\{\Delta_{13}, \Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34}\}$$

is a positivity test.

Question 1.8. How can one find positivity tests systematically?

For $\mathrm{Gr}(2,4)$, another positivity test is

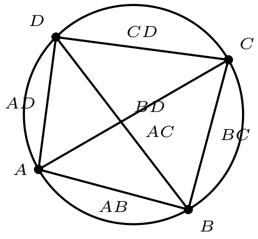
$$\{\Delta_{24}, \Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34}\}.$$



1.3.2 Ptolemy's relation and positivity tests

The Plücker relation for $\mathrm{Gr}(2,4)$ has the same form as Ptolemy's relation for a cyclic quadrilateral.

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD}.$$



Remark 1.9. For $\mathrm{Gr}(2,n)$, Plücker coordinates are naturally indexed by diagonals and boundary edges of an n-gon. A triangulation gives a collection of n boundary edges together with $n-3$ diagonals. This collection forms a cluster. Flipping one diagonal in a quadrilateral corresponds algebraically to replacing one Plücker coordinate by another using a Ptolemy-type relation.

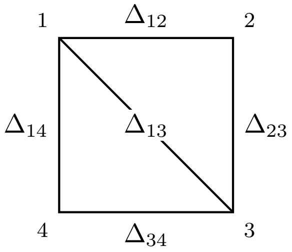
Thus, in this setting, one has the guiding correspondence

$$\left\{ \begin{array}{c} \text{efficient} \\ \text{positivity tests} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of the } n\text{-gon} \end{array} \right\}.$$

1.4 Triangulation Cluster Algebras

1.4.1 The algebra from a square

Let T be the triangulation of the square using the diagonal 13.



Definition 1.10 (Cluster algebra attached to T)**.** Let T be the triangulation above. The associated cluster algebra $\mathcal{A}_T$ is the subalgebra of

$$\mathbb{Q}(\Delta_{13}, \Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34})$$

generated by all cluster variables obtained by iterated mutation. Here:

$$\mathcal{A}_T = \left\langle \Delta_{13}, \Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34}, \frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}}{\Delta_{13}} \right\rangle.$$

Because

$$\frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}}{\Delta_{13}} = \Delta_{24},$$

we have

$$\mathcal{A}_T = \langle \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34} \rangle .$$

Remark 1.11. More precisely, the homogeneous coordinate ring of $\mathrm{Gr}(2,4)$ in its Plücker embedding is

$$\mathbb{K}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}] \Big/ \Big(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}\Big).$$

Recall that the Grassmannian is the variety of k-dimensional subspaces of a vector space of dimension n:

$$\mathrm{Gr}_k(n) := \{\, V \subseteq \mathbb{K}^n : \dim_{\mathbb{K}}(V) = k \,\}.$$

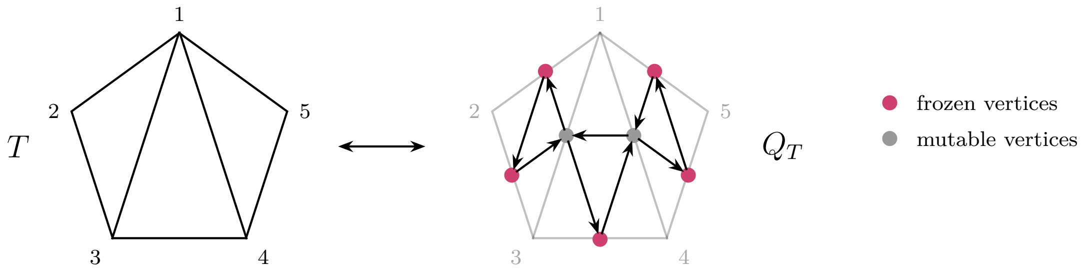
Theorem 1.12 (Scott, 2006)**.** *The homogeneous coordinate ring of the Grassmannian* $\mathrm{Gr}_k(n)$ *in its Plücker embedding admits a cluster algebra structure.*

Remark 1.13. For $\mathrm{Gr}(2,n)$, this cluster structure is especially concrete: clusters are indexed by triangulations of an n-gon, cluster variables are Plücker coordinates, and mutation is diagonal flip.

1.4.2 Quivers and Mutation

1.4.3 From triangulations to quivers

Given a triangulation T of a polygon, one associates a quiver Q_T. Boundary edges produce frozen vertices, and diagonals produce mutable vertices.



Definition 1.14 (Quiver)**.** A *quiver* is a directed graph. In the cluster algebra setting considered here, we assume that quivers have no loops and no oriented 2-cycles.

Definition 1.15 (Quiver mutation)**.** Let Q be a quiver without loops or oriented 2-cycles, and let k be a mutable vertex of Q. The mutation $\mu_k(Q)$ is obtained by the following three-step procedure:

(1) for every oriented path $i \to k \to j$, add an arrow $i \to j$;

(2) reverse every arrow incident to k;

(3) remove all oriented 2-cycles.

1.4.4 The exchange relation

The Plücker relation

$$\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$$

is a special case of the general *exchange relation* that governs cluster mutation:

$$x_k' x_k = \prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j.$$

Definition 1.16 (Seed)**.** A *seed* is a pair

$$s = (\mathbf{x}, Q),$$

where Q is a quiver and $\mathbf{x}$ is a collection of algebraically independent variables assigned to the vertices of Q.

Definition 1.17 (Seed mutation)**.** Let $s = (\mathbf{x}, Q)$ be a seed and let k be a mutable vertex of Q. The mutation of s in direction k is

$$\mu_k(s) = \big(\mu_k(\mathbf{x}), \mu_k(Q)\big),$$

where the new cluster is

$$\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x_k'\},$$

with x_k' determined by the exchange relation

$$x_k' x_k = \prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j.$$

Remark 1.18. The variables at mutable vertices change under mutation. The variables at frozen vertices do not mutate; they behave like coefficients. In the polygon model for $\mathrm{Gr}(2, n)$, boundary edges correspond to frozen variables and diagonals correspond to mutable variables.

1.5 Example: Seed Mutation in Type A_2

Consider the initial quiver

$$x_1 \longrightarrow x_2.$$

Mutation involves the exchange relation

$$x_k' x_k = \prod_{i \to k} x_i + \prod_{k \to j} x_j.$$

Starting from Q, repeated mutation produces the following five cluster variables:

$$x_1, \qquad x_2, \qquad x_3 := \frac{1+x_2}{x_1}, \qquad x_4 := \frac{1+x_1+x_2}{x_1x_2}, \qquad x_5 := \frac{1+x_1}{x_2}.$$

$$\begin{tikzcd}
 & & x_1 \arrow[r] \arrow[dll, "1"'] & x_2 \arrow[dr, leftrightarrow, "\cong"] & & \\
x_3 := \frac{1+x_2}{x_1} \arrow[dr, "2"'] & x_2 \arrow[l] & & & x_2 \arrow[d, "1"] & x_1 \arrow[l] \\
\frac{1+x_2}{x_1} \arrow[r] & \frac{1+x_1+x_2}{x_1x_2} =: x_4 \arrow[dr, "1"'] & & & \frac{1+x_1}{x_2} \arrow[r] \arrow[dl, "2"] & x_1 \\
 & & x_5 := \frac{1+x_1}{x_2} & \frac{1+x_1+x_2}{x_1x_2} \arrow[l] & &
\end{tikzcd}$$

Definition 1.19 (The cluster algebra of the quiver A_2)**.** For the quiver Q, the associated cluster algebra is

$$\mathcal{A}_Q := \left\langle x_1,\, x_2,\, \frac{1+x_2}{x_1},\, \frac{1+x_1+x_2}{x_1x_2},\, \frac{1+x_1}{x_2} \right\rangle \subseteq \mathbb{Q}(x_1, x_2).$$

Remark 1.20 (Added Context)**.** This is the cluster algebra of finite type A_2. The five cluster variables correspond combinatorially to the five diagonals of a pentagon.

1.6 Fundamental Theorems

Theorem 1.21 (Laurent phenomenon, Fomin–Zelevinsky 2001)**.** *Every cluster variable is a Laurent polynomial in the variables of any fixed initial cluster, with integer coefficients. More precisely, if the initial seed has mutable variables*

$$x_1, \dots, x_n$$

and frozen variables

$$x_{n+1}, \dots, x_{n+m},$$

then all cluster variables belong to the ring

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{n+m}].$$

Remark 1.22. The Laurent phenomenon is highly nontrivial because mutation is defined by division. A priori, repeated mutation could produce arbitrary rational functions. The theorem says that all denominators remain monomials in the initial mutable variables.

Theorem 1.23 (Positivity conjecture, Fomin–Zelevinsky 2001)**.** *Every cluster variable is a Laurent polynomial with nonnegative integer coefficients in the variables of any fixed initial cluster:*

$$\textit{cluster variables} \subseteq \mathbb{N}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{n+m}].$$

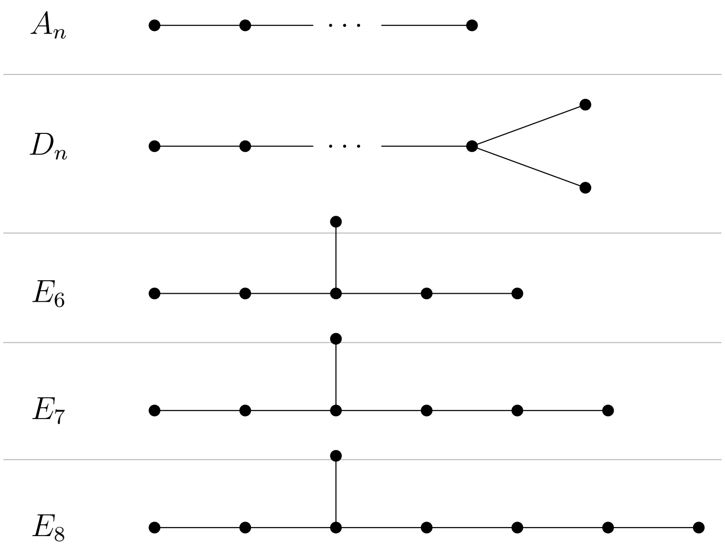
Theorem 1.24 (Positivity theorem, Gross–Hacking–Keel–Kontsevich)**.** *The positivity conjecture is true.*

Remark 1.25. Gross–Hacking–Keel–Kontsevich view cluster algebras as rings of functions on log Calabi–Yau varieties, also called cluster varieties, and use tools from birational geometry and mirror symmetry.

1.7 Finite Type Classification

Definition 1.26 (Finite type)**.** A cluster algebra is said to be of *finite type* if it has only finitely many seeds, equivalently only finitely many cluster variables.

Theorem 1.27 (Fomin–Zelevinsky, 2003)**.** *Let $\mathcal{A}$ be a cluster algebra associated to a skew-symmetric quiver Q. Then $\mathcal{A}$ is of finite type if and only if the mutable part of Q is mutation equivalent to an orientation of a simply-laced Dynkin diagram of type A_n, D_n, or E_6, E_7, E_8.*



Remark 1.28. The theorem establishes a deep connection between cluster algebras and classical Lie theory. Mutation equivalence of quivers is governed by the same Dynkin diagrams that classify simply-laced finite root systems.

1.8 Examples of Cluster Structures

1.8.1 Markov triples

Definition 1.29 (Markov triple)**.** A *Markov triple* is a triple of integers (a, b, c) satisfying

$$a^2 + b^2 + c^2 = 3abc.$$

Consider the quiver

$$Q = \begin{tikzcd} & 2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

which comes from a torus with one puncture. It is mutation equivalent to itself. An initial Markov triple is

$$(x_1, x_2, x_3) = (1, 1, 1).$$

If

$$(\mathbf{x}', Q') \sim (\mathbf{x}, Q)$$

are mutation equivalent, then

$$\mathbf{x}' = (x_1', x_2', x_3')$$

is another Markov triple.

Remark 1.30. For the Markov quiver, mutation at one vertex replaces one component of the triple by

$$a' = 3bc - a,$$

which is exactly the classic method for generating new Markov numbers from old ones.

1.8.2 The Jones polynomial

Cluster structures also appear in knot theory.

$$\begin{array}{ccccccc} \text{knot or link} & & \text{continued} & & \text{snake} & & \text{cluster} \\ \text{of two bridges } L & \rightsquigarrow & \text{fraction} & \rightsquigarrow & \text{graph} & \rightsquigarrow & \text{variable } x_L. \end{array}$$

Theorem 1.31 (Lee–Schiffler)**.** *The Jones polynomial of a two-bridge knot or link L, up to normalization by the leading term, is equal to the Laurent polynomial of the associated cluster variable x_L, up to a specialization of coefficients.*

1.8.3 Scattering amplitudes

Cluster algebras also appear in the study of scattering amplitudes in quantum field theory. This connection is developed in more detail in the next sections.

1.9 Cluster Algebras and Scattering Amplitudes

1.9.1 Physical motivation

In quantum field theory, physicists compute probabilities of particle interactions via scattering amplitudes.

A *scattering amplitude* is a function on a kinematic space that models the particle configuration. Traditionally, amplitudes are computed as infinite sums of Feynman integrals, which are usually very difficult to evaluate.

More recent approaches study scattering amplitudes in complex coordinate systems, including twistor-theoretic coordinates. These methods express amplitudes as iterated integrals and analyze:

- their singularity structure, encoded in their *symbol*;

- recursive structures of amplitudes;
- geometric interpretations, for example as volumes of the *amplituhedron*.

Remark 1.32. Cluster variables often provide natural coordinates for the singularities of scattering amplitudes. In this way, cluster combinatorics can constrain which logarithmic singularities are allowed to appear.

1.10 Configurations to Varieties

1.10.1 Minkowski space and helicity spinors

Let $\mathbb{R}^{1,3}$ be Minkowski space with inner product

$$p \cdot p' = p_0 p_0' - (p_1 p_1' + p_2 p_2' + p_3 p_3').$$

The map σ is the standard identification between Minkowski spacetime $\mathbb{R}^{1,3}$ and the space of 2×2 Hermitian matrices.

$$\sigma(p) = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

For a lightlike vector p with $p^2 = 0$, define

$$\lambda = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 \\ p_1 + ip_2 \end{pmatrix}, \qquad \widetilde{\lambda} = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 \\ p_1 - ip_2 \end{pmatrix}.$$

Then

$$\sigma(p) = \lambda \widetilde{\lambda}^T.$$

Remark 1.33. The map σ is the standard identification between Minkowski spacetime $\mathbb{R}^{1,3}$ and the space of 2×2 Hermitian matrices. This is the basic bridge between physical momentum space and spinor-helicity coordinates.

The mass m of p satisfies

$$p^2 := p \cdot p = m^2.$$

A particle is massless if and only if $m = 0$, which is equivalent to

$$\det(\sigma(p)) = p_0^2 - (p_1^2 + p_2^2 + p_3^2) = p^2 = 0.$$

1.10.2 The spinor-helicity variety

Let

$$\{p_1, \dots, p_n\} \subseteq \mathbb{R}^4$$

be the momenta of n interacting massless particles. By momentum conservation,

$$\sum_{i=1}^{n} p_i = 0.$$

Each momentum p_i is represented by a pair of spinors

$$(\lambda_i, \widetilde{\lambda}_i) \in \mathbb{C}^2 \times \mathbb{C}^2.$$

Collect these spinors into two $2 \times n$ matrices:

$$\Lambda = [\lambda_1, \dots, \lambda_n], \qquad \widetilde{\Lambda} = [\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_n].$$

Exercise 1.34. Momentum conservation translates into

$$\Lambda \widetilde{\Lambda}^T = 0.$$

Proof. For each particle i, the spinor-helicity factorization gives

$$\sigma(p_i) = \lambda_i \widetilde{\lambda}_i^T.$$

Summing over all particles, we get

$$\sigma\left(\sum_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} \sigma(p_i) = \sum_{i=1}^{n} \lambda_i \widetilde{\lambda}_i^T = \Lambda \widetilde{\Lambda}^T.$$

Since total momentum is zero, the result follows. □

Definition 1.35 (Spinor-helicity variety)**.** The *spinor-helicity variety* is

$$\mathcal{SH}_n := \{(\Lambda, \widetilde{\Lambda}) \in \mathrm{Gr}(2,n) \times \mathrm{Gr}(2,n) : \Lambda \widetilde{\Lambda}^T = 0\}.$$

The variety $\mathcal{SH}_n$ is parametrized by $\binom{n}{2}$ minors

$$P_{ij} := \det(\lambda_i, \lambda_j), \qquad \widetilde{P}_{ij} := \det(\widetilde{\lambda}_i, \widetilde{\lambda}_j), \qquad 1 \le i < j \le n.$$

These coordinates satisfy the Plücker relations for both Grassmannians:

$$P_{ij} P_{kl} - P_{ik} P_{jl} + P_{il} P_{jk} = 0$$

$$\widetilde{P}_{ij}\widetilde{P}_{kl} - \widetilde{P}_{ik}\widetilde{P}_{jl} + \widetilde{P}_{il}\widetilde{P}_{jk} = 0,$$

and the conditions derived from momentum conservation:

$$\sum_{s=1}^{n} P_{is}\widetilde{P}_{sj} = 0,$$

which is the coordinate-level expression of

$$\Lambda\widetilde{\Lambda}^T = 0.$$

Remark 1.36. The determinantal identities

$$P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} = 0$$

and their duals define the ambient product of Grassmannians.

Remark 1.37. The spinor-helicity variety is therefore a subvariety of a product of two Grassmannians cut out by equations expressing conservation of total momentum. Since Grassmannian coordinate rings carry cluster structures, it is natural to ask whether the coordinate ring of $\mathcal{SH}_n$ also has a cluster algebra structure.

1.11 Spinor-Helicity Varieties

Theorem 1.38 (Bossinger–Jianrong Li)**.** *The homogeneous coordinate ring of the spinor-helicity variety is a cluster algebra that embeds into a Grassmannian cluster algebra.*

Theorem 1.39 (Bossinger–Drummond-Glew; Bossinger–Drummond-Glew–Gürdoğan –Wright; Pokraka–Spradlin–Volovich–Weng)**.** *For five or fewer particles, the scattering amplitude is uniquely determined by the cluster algebra. For six particles, the symbol is partially given by cluster variables.*

Theorem 1.40 (Galashin)**.** *The real spinor-helicity variety contains a positive subset, namely the momentum amplituhedron, equipped with a canonical volume form whose integral is the scattering amplitude.*

Remark 1.41. These results illustrate the central theme of the lecture: cluster algebras provide a common language linking total positivity, Grassmannians, algebraic geometry, and scattering amplitudes.

1.12 The Symbol of a Polylogarithm

Example 1.42 (Polylogarithms)**.** For $k \geq 1$, the polylogarithm is

$$\mathrm{Li}_k(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^k}.$$

It satisfies the recursive integral representation

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t)\, d\log t, \qquad \mathrm{Li}_1(z) = -\log(1-z).$$

Its symbol is defined iteratively:

$$\mathcal{S}(\mathrm{Li}_1(z)) = -(1-z),$$

and for higher weights,

$$\mathcal{S}(\mathrm{Li}_k(z)) = -(1-z) \otimes z^{\otimes(k-1)}.$$

Remark 1.43. The symbol map $\mathcal{S}$ sends iterated integrals, modulo functions of lower transcendentality, into a tensor algebra. It is an algebraic invariant that simplifies the study of polylogarithmic identities, especially those appearing in quantum field theory.

Remark 1.44. In the context of scattering amplitudes, the entries appearing in symbols are often expected to come from distinguished algebraic functions on the relevant kinematic variety. Cluster variables provide a natural and highly structured source for such functions.

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