

KUMUNU 2026

Commutative Algebra and Algebraic Geometry Conference

University of Kansas

Lawrence, Kansas

October 10–11, 2026

Conference Notes

KUMUNU 2026

Named for its original participating institutions—the University of Kansas, the University of Missouri–Columbia, and the University of Nebraska–Lincoln—the conference has been held annually since 1999. It provides a setting to share cutting-edge mathematical research and promote collaborations among algebraists in and near the Great Plains region.

Notes Prepared by Soumyadeep Misra (Unofficial). These notes were created with the assistance of AI. The author takes full responsibility for any mistakes.

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Tor Rigidity and Frobenius

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Saturday, October 10, 2026 ■ 330 Strong Hall

Abstract

The Frobenius endomorphism of a local ring in characteristic p is closely related to the vanishing of higher Tor and the detection of finite homological dimension. These notes review Kunz’s characterization of regularity, the theorems of Peskine–Szpiro and Herzog, rigidity over complete intersections, and examples of nonrigidity. A more recent derived-category approach uses codepth and Frobenius pushforwards of Koszul complexes to obtain effective vanishing criteria.

1 The Frobenius module and its homological functors

Throughout, $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring of characteristic $p > 0$, and $q = p^e$ for a positive integer e . The e th iterate of Frobenius is

$$F^e: R \longrightarrow R, \quad r \longmapsto r^q.$$

The notation ${}^eR$ denotes the additive group of R with its R -module structure restricted along F^e :

$$r \cdot s = r^q s \quad (r \in R, s \in {}^eR). \quad (1.1)$$

When it is useful to emphasize the functor, we write $F_*^e N$ for a module N restricted along F^e ; thus $F_*^e R = {}^eR$. In contrast, the *Frobenius functor* is extension of scalars along F^e ,

$$\mathcal{F}_R^e(M) = M \otimes_R {}^eR, \quad (1.2)$$

where R acts on the resulting module by ordinary multiplication on the right factor. The underlying tensor product in (1.2) must not be confused with restriction of scalars $F_*^e M$. The former is right exact; the latter is exact as a functor on underlying abelian groups.

Theorem 1.1 (Kunz, 1969). *The ring R is regular if and only if ${}^eR$ is a flat R -module for some positive integer e , equivalently for every positive integer e .*

In particular, regularity can be characterized by the vanishing of the higher derived functors of Frobenius base change for *all* modules. The interesting rigidity question is different: for a

fixed module M , how much vanishing of

$$\mathrm{Tor}_i^R(M, {}^eR) \quad (i > 0)$$

is required to conclude that M has finite projective or flat dimension?

2 Classical vanishing and rigidity

2.1 Finite projective dimension and the converse problem

Theorem 2.1 (Peskin–Szpiro). *If M is a finitely generated R -module of finite projective dimension, then*

$$\mathrm{Tor}_i^R(M, {}^eR) = 0 \quad \text{for every } i > 0 \text{ and } e > 0.$$

Moreover, $\mathrm{pd}_R \mathcal{F}_R^e(M) < \infty$ for each $e > 0$.

Remark 2.2. The vanishing is special to the Frobenius endomorphism: a module of finite projective dimension need not have vanishing higher Tor with an arbitrary second module. Peskin–Szpiro prove acyclicity of Frobenius on finite free resolutions of modules of finite projective dimension. The conclusion concerning $\mathcal{F}_R^e(M)$ follows from that acyclicity.

The converse requires more care. In particular, a vanishing condition involving a single homological degree should not be replaced without justification by a condition involving all positive degrees.

Theorem 2.3 (Herzog, 1974). *For a finitely generated R -module M , the following are equivalent:*

- (i) $\mathrm{pd}_R M < \infty$;
- (ii) $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for all $i > 0$ and all $e > 0$;
- (iii) $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for all $i > 0$ and infinitely many e .

The implication (i) $\Rightarrow$ (ii) is Theorem 2.1. Herzog’s converse shows that the family of Frobenius twists detects finite projective dimension; it does not assert that one arbitrary isolated vanishing of Tor suffices.

Theorem 2.4 (Koh–Lee, 1998). *There exists an integer $c(R)$ with the following property.*

If M is a finitely generated R -module and, for some $e \geq c(R)$, the modules

$$\mathrm{Tor}_t^R(M, {}^eR), \dots, \mathrm{Tor}_{t+\mathrm{depth} R}^R(M, {}^eR)$$

vanish for a positive integer t , then $\mathrm{pd}_R M < \infty$.

Theorem 2.5 (Claudia Miller, 2003 [33]). *If R is Cohen–Macaulay of positive dimension, then, for a sufficiently large Frobenius exponent e , $\dim R = \mathrm{depth} R$ consecutive positive vanishing degrees suffice to conclude that a finitely generated R -module M has finite projective dimension.*

Thus there is a ring-dependent threshold on the Frobenius exponent, as well as a requirement on the *number of consecutive* vanishing degrees.

2.2 A single vanishing over complete intersections

Theorem 2.6 (Avramov–Miller; Dutta). *Suppose R is a local complete intersection and M is a finitely generated R -module. If*

$$\mathrm{Tor}_i^R(M, {}^eR) = 0$$

for one pair of positive integers (i, e) , then $\mathrm{pd}_R M < \infty$.

Combining this theorem with Theorem 2.1, one obtains vanishing in every positive degree for every Frobenius iterate. Such a consequence is much stronger than the vanishing in a consecutive finite range in Theorem 2.4.

Definition 2.7 (Rigidity and strong rigidity). Let N be an R -module, and fix a class $\mathcal{C}$ of finitely generated R -modules.

(i) N is *rigid on $\mathcal{C}$* if for every $M \in \mathcal{C}$ and $i > 0$,

$$\mathrm{Tor}_i^R(M, N) = 0 \implies \mathrm{Tor}_j^R(M, N) = 0 \text{ for every } j \geq i.$$

(ii) N is *strongly rigid on $\mathcal{C}$* if for every $M \in \mathcal{C}$ and $i > 0$,

$$\mathrm{Tor}_i^R(M, N) = 0 \implies \mathrm{pd}_R M < \infty.$$

Without qualification, the class is all finitely generated modules.

For $N = {}^eR$, strong rigidity implies rigidity by Theorem 2.1. In view of Theorem 2.6, every

Frobenius twist of a local complete intersection is strongly rigid. These notions and their limitations were studied systematically by Dao–Li–Miller [20].

2.3 A determinantal counterexample

The complete-intersection hypothesis cannot simply be omitted. The lecture records the determinantal construction from Dao–Li–Miller, for which the choice of module is important.

Example 2.8 (Dao–Li–Miller). Let K be a field of characteristic $p > 0$ and $X = (X_{ij})$ a generic 3×3 matrix. Put

$$A = K[X_{ij} : 1 \leq i, j \leq 3]/I_2(X), \quad R = A_{(x_{ij})}, \quad L = R/(x_{11}, x_{12}),$$

where $I_2(X)$ is the ideal of all 2×2 minors and x_{ij} denotes the residue class of X_{ij} . Then

$$\mathrm{pd}_R L = \infty, \quad \mathrm{depth}_R \mathcal{F}_R^e(L) > 0 \quad \text{for every } e > 0. \quad (2.1)$$

Dao–Li–Miller use L to construct a different finitely generated module N of infinite projective dimension satisfying, for every $e > 0$,

$$\mathrm{Tor}_1^R(N, {}^eR) = 0, \quad \mathrm{Tor}_2^R(N, {}^eR) \neq 0. \quad (2.2)$$

Consequently ${}^eR$ is not rigid over R , for any $e > 0$.

Remark 2.9. It would be incorrect to substitute L for N in (2.2). The explicitly displayed cyclic module L has property (2.1), whereas the simultaneous assertions about Tor_1 and Tor_2 belong to the auxiliary module N constructed in the proof of the cited result [20, Lemma 3.4, Example 3.5 and Remark 3.6]. This distinction repairs a compression in the handwritten account.

3 A simple Artinian example and arbitrary modules

The failure of complete-intersection structure does not by itself prevent strong rigidity. The following example, recorded on the second page, makes this clear.

Example 3.1. Let k be a field of characteristic $p > 0$ and

$$R = k[[x, y]]/(x, y)^2, \quad \mathfrak{m} = (\bar{x}, \bar{y}).$$

Then R is Artinian and not a complete intersection, since $(x, y)^2$ requires three generators

in the regular local ring $k[[x, y]]$. For every $e \geq 1$,

$$\mathfrak{m}^{[p^e]} = (0), \quad \mathfrak{m} \cdot {}^eR = 0.$$

In particular, ${}^eR$ is a nonzero k -vector space as an R -module, hence a direct sum of copies of k ; the number of copies need not be finite unless the relevant Frobenius field extension is finite. For each finitely generated R -module M and $i > 0$ we therefore have

$$\mathrm{Tor}_i^R({}^eR, M) = 0 \implies \mathrm{Tor}_i^R(k, M) = 0.$$

A minimal free resolution of M gives $\mathrm{Tor}_i^R(k, M) \cong k^{\beta_i^R(M)}$; hence vanishing for a single $i > 0$ forces $\mathrm{pd}_R M < i$. Thus each ${}^eR$ is strongly rigid, although R is not a complete intersection.

For non-finitely generated modules, the distinction between finite *projective* dimension and finite *flat* dimension matters. The converse to Frobenius acyclicity extends to arbitrary modules in the following form.

Theorem 3.2 (Marley–Webb; Dailey–Iyengar–Marley). *Let M be an arbitrary R -module. The following are equivalent:*

- (i) $\mathrm{fd}_R M < \infty$;
- (ii) $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for all $i > 0$ and all $e > 0$;
- (iii) *there is an integer $t > 0$ such that $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for $t \leq i \leq t + \dim R$ and for infinitely many e .*

If R is Cohen–Macaulay of multiplicity $e(R)$, then in (iii) one $e > \log_p e(R)$ is enough.

Remark 3.3. The equivalence involving vanishing in all positive degrees and arbitrary modules is due to Marley–Webb [34]. The bounded-range result and its Cohen–Macaulay refinement are proved for modules, indeed for suitable complexes, in [19]. The replacement of projective by flat dimension here is essential.

Theorem 3.4 (Funk–Marley [24, 25]). *Suppose $d = \dim R > 0$ and let M be an arbitrary R -module. If there is an integer $t > 0$ such that*

$$\mathrm{Tor}_i^R(M, {}^eR) = 0 \quad (t \leq i \leq t + d - 1)$$

for infinitely many e , then $\mathrm{fd}_R M < \infty$. If R is Cohen–Macaulay, the same conclusion follows from this block of d vanishings for a single $e > \log_p e(R)$.

This strengthens the bounded-range criterion in Theorem 3.2 from $d + 1$ to d consecutive

degrees when $d > 0$. The corrected proof of the Cohen–Macaulay case is given in [25].

4 Codepth and the derived-category approach

We replace a ring-dependent bound on the Frobenius iterate by a more explicit invariant. Set

$$b = \operatorname{edim} R = \dim_k \mathfrak{m}/\mathfrak{m}^2, \quad c = \operatorname{codepth} R = b - \operatorname{depth} R. \quad (4.1)$$

If $R \cong Q/I$ is a minimal Cohen presentation of a complete local ring, with $(Q, \mathfrak{n})$ regular local and $I \subseteq \mathfrak{n}^2$, the Auslander–Buchsbaum formula yields

$$\operatorname{codepth} R = \operatorname{pd}_Q R.$$

More generally $\operatorname{codepth} R_{\mathfrak{p}} \leq \operatorname{codepth} R$ for each $\mathfrak{p} \in \operatorname{Spec} R$. This localization inequality is useful when proving finite-flat-dimension criteria for arbitrary modules [10].

4.1 Koszul complexes after Frobenius restriction

Choose a minimal generating sequence $\mathbf{x} = x_1, \dots, x_b$ of $\mathfrak{m}$ and let $K = K^R(\mathbf{x})$ be the Koszul complex, supported in homological degrees $0, \dots, b$. In particular,

$$H_0(K) = k, \quad H_i(K) = 0 \quad (i > c).$$

The next theorem is a special case of the local form of the work of Ballard–Iyengar–Lank–Mukhopadhyay–Pollitz [10, Theorem 2.1].

Theorem 4.1 (Ballard–Iyengar–Lank–Mukhopadhyay–Pollitz). *For any R -complex C and positive integer e satisfying*

$$p^e > \operatorname{codepth} R,$$

there is an isomorphism in $\mathbf{D}(R)$

$$F_*^e(K \otimes_R C) \simeq \bigoplus_{i \in \mathbb{Z}} \Sigma^i H_i \left(F_*^e(K \otimes_R C) \right). \quad (4.2)$$

In particular, k is a direct summand of $F_*^e K$ in $\mathbf{D}(R)$.

Explanation of the direct-summand consequence. Take $C = R$. The displayed decomposition splits off the degree-zero homology $H_0(F_*^e K) = F_*^e H_0(K) = F_*^e k$. The latter is a nonzero k -vector space annihilated by $\mathfrak{m}$, and therefore contains k as an R -module direct summand. Thus k is also a summand of the original complex $F_*^e K$ in $\mathbf{D}(R)$. The formality isomorphism itself is the deep content of [10, Theorem 2.1] and is not being reproved here. $\square$

Remark 4.2. The hypothesis is $p^e > c$, equivalently $e > \log_p c$ when $c > 0$. Writing $p^e > c$ also covers $c = 0$ without taking a logarithm of zero. The direct-summand assertion is in the derived category: it does not claim that k is a direct summand of the module ${}^eR$.

Theorem 4.3 (Derived generation, F -finite case). *Assume in addition that R is F -finite, so that ${}^eR$ is finitely generated. If $p^e > \text{codepth } R$, then ${}^eR$ is a strong generator of $D^b(\text{mod } R)$.*

Theorem 4.3 is a categorical assertion: every object of $D^b(\text{mod } R)$ can be built from ${}^eR$ using shifts, finite sums, summands and a uniformly bounded number of cones. This *strong generation* should not be confused with *strong rigidity* in Definition 2.7.

4.2 An effective consecutive-vanishing criterion

Proposition 4.4 (Consequence for finitely generated modules). *Let M be a finitely generated R -module. If $p^e > \text{codepth } R$ and, for some $t > 0$,*

$$\text{Tor}_i^R({}^eR, M) = 0 \quad (t \leq i \leq t + b), \quad b = \text{edim } R, \quad (4.3)$$

then $\text{pd}_R M < \infty$; indeed $\text{pd}_R M < t + b$.

Proof. As a complex of R -modules, $F_*^e K$ has one term isomorphic to a finite direct sum of ${}^eR$ in each degree $0, \dots, b$. There is a bounded hyperhomology spectral sequence

$$E_{u,v}^1 = \text{Tor}_v^R(F_*^e K_u, M) \implies H_{u+v}((F_*^e K) \otimes_R^{\mathbf{L}} M).$$

On the total-degree diagonal $u + v = t + b$ one has $v = t + b - u$, hence $t \leq v \leq t + b$. Every term of this diagonal vanishes by (4.3); therefore

$$H_{t+b}((F_*^e K) \otimes_R^{\mathbf{L}} M) = 0.$$

By Theorem 4.1, the residue field k is a direct summand of $F_*^e K$ in the derived category. Consequently $\text{Tor}_{t+b}^R(k, M) = 0$. Since M is finitely generated, its minimal free resolution satisfies $\text{Tor}_{t+b}^R(k, M) \cong k^{\beta_{t+b}^R(M)}$. Thus $\beta_{t+b}^R(M) = 0$, and minimality implies $\text{pd}_R M < t + b$. $\square$

Remark 4.5 (Arbitrary modules and localization). We also have an analogous finite-*flat*-dimension statement for arbitrary M , with $b + 1$ consecutive vanishing degrees and the

same exponent bound. The intended reduction involves localizing at each $\mathfrak{p}$, using

$$\text{codepth } R_{\mathfrak{p}} \leq \text{codepth } R, \quad \text{edim } R_{\mathfrak{p}} \leq \text{edim } R,$$

and detecting flat dimension from Tor with residue fields of the localizations. Since the handwritten record does not supply the final arbitrary-module detection argument, Proposition 4.4 is stated and proved only in the finitely generated case. The stronger formulation is retained here as a lecture-reported extension, not as an independently proved theorem.

5 Modules of finite complete intersection dimension

For a finitely generated R -module M , recall that a *quasi-deformation* consists of local homomorphisms

$$R \longrightarrow R' \longleftarrow Q,$$

where $R \rightarrow R'$ is flat and $Q \rightarrow R'$ has kernel generated by a Q -regular sequence. Its complete intersection dimension is

$$\text{CI-dim}_R(M) = \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q R'\},$$

with the infimum taken over such diagrams for which these quantities are defined. A local complete intersection has finite CI-dimension for all finitely generated modules, whereas the condition $\text{CI-dim}_R(M) < \infty$ can hold for an individual module over a ring that is not a complete intersection.

Theorem 5.1 (Announced in the lecture). *Let M be a finitely generated R -module with $\text{CI-dim}_R(M) < \infty$. Then for a fixed $e > 0$:*

$$\text{Tor}_i^R({}^eR, M) = 0 \text{ for some } i > 0 \implies \text{Tor}_j^R({}^eR, M) = 0 \text{ for all } j \gg 0. \quad (5.1)$$

Corollary 5.2 (Conditional deduction from Theorem 5.1). *Assuming the announced implication (5.1), if M is finitely generated with $\text{CI-dim}_R(M) < \infty$ and $p^e > \text{codepth } R$, then*

$$\text{Tor}_i^R({}^eR, M) = 0 \text{ for some } i > 0 \implies \text{pd}_R M < \infty.$$

Thus, subject to the announced theorem, ${}^eR$ is strongly rigid on the class of finitely generated modules of finite CI-dimension whenever $p^e > \text{codepth } R$.

Proof. By (5.1), there is an integer $t > 0$ such that $\text{Tor}_j^R({}^eR, M) = 0$ for all $j \geq t$. In particular this holds in the $b + 1$ consecutive degrees $t, \dots, t + b$, where $b = \text{edim } R$. Proposition 4.4 then gives finite projective dimension. $\square$

Carlsson's Conjecture and the Tate-valued Frobenius

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KUMUNU 2026 ■ October 11, 2026

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Abstract

Rank conjectures are ubiquitous in commutative algebra and topology. At a conceptual level, they predict that objects constrained by many independent equations, or endowed with many independent symmetries, must have "large" derived invariants. In 2018, Iyengar and Walker showed that many of these rank conjectures are false in odd characteristic, leaving the status unknown but precarious in characteristic 2. In this talk, I will explain recent work showing that the algebraic and topological forms of these conjectures are true in characteristic 2. The key idea is to use the so-called Tate construction as an intermediary to compare tensor and Frobenius powers.

6 Carlsson's conjecture and total-rank questions

One can ask the question, how large an elementary abelian 2-group can act freely on a product of spheres? The case of a single sphere is governed by classical restrictions on free involutions, and the cases of one and two sphere factors were associated in the notes with Smith, Conner, and Heller [38, 17, 26]. The general problem is often called the *sphere-rank conjecture*, now proved for elementary abelian 2-groups by the characteristic-two result in [39]. The next two statements retain their historical conjecture labels; their proved forms appear in [corollaries 10.1](#) and [10.2](#).

Conjecture 6.1 (Former sphere-rank conjecture; proved for elementary abelian 2-groups). *Let $E = (\mathbb{Z}/2\mathbb{Z})^r$ act freely on a finite CW complex*

$$X \simeq S^{n_1} \times \cdots \times S^{n_m}, \quad n_j \geq 1.$$

Then $r \leq m$.

The stronger assertion proposed by Carlsson concerns an arbitrary finite free E -complex, without requiring the homotopy type of a product of spheres.

Conjecture 6.2 (Carlsson, 1986; former conjecture, proved in characteristic two). *Let X be a nonempty finite free E -CW complex, where $E = (\mathbb{Z}/2\mathbb{Z})^r$. Then*

$$\sum_{i \geq 0} \dim_{\mathbb{F}_2} H_i(X; \mathbb{F}_2) \geq 2^r. \quad (6.1)$$

Carlsson had established this inequality for $r \leq 3$; see [16]. It immediately implies the sphere-rank inequality, since the Künneth theorem over $\mathbb{F}_2$ gives

$$\sum_i \dim_{\mathbb{F}_2} H_i(S^{n_1} \times \cdots \times S^{n_m}; \mathbb{F}_2) = 2^m.$$

Consequently, a free E -action as in ?? 6.1 would force $2^r \leq 2^m$, hence $r \leq m$.

6.1 Algebraic formulations

Let

$$S = \mathbb{F}_2[x_1, \dots, x_r], \quad |x_j| = -1.$$

Here the grading is homological, and the differential has degree -1 . A finite free dg S -module means one which is free of finite rank as a graded S -module.

Theorem 6.3 (Characteristic-two algebraic Carlsson inequality [39]). *Let P be a finite free dg S -module such that*

$$0 < \sum_i \dim_{\mathbb{F}_2} H_i(P) < \infty.$$

Then

$$\text{rank}_S(P) \geq 2^r. \quad (6.2)$$

The nonvanishing assumption cannot be discarded: a contractible rank-two dg module has zero homology regardless of r . The passage from the group-action problem to theorem 6.3 uses Carlsson's Koszul duality and, in the topological setting, an equivariant cochain construction; it is not a formal consequence of a numerical coincidence [15, 39].

6.2 Buchsbaum–Eisenbud–Horrocks and generalized total rank

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d , and let $M \neq 0$ be a finitely generated R -module of finite length and finite projective dimension. Write

$$\beta_i^R(M) = \dim_k \text{Tor}_i^R(M, k).$$

Conjecture 6.4 (Buchsbaum–Eisenbud–Horrocks). *Under the preceding assumptions,*

$$\beta_i^R(M) \geq \binom{d}{i} \quad (0 \leq i \leq d).$$

The weaker *total rank conjecture* predicts

$$\sum_{i \geq 0} \beta_i^R(M) \geq 2^d. \quad (6.3)$$

The finite-projective-dimension hypothesis makes the Betti sum finite. The inequalities in 6.4 imply (6.3), since $\sum_i \binom{d}{i} = 2^d$. For a minimal free resolution, the sum is its total free rank. For relevant historical developments see [1, 42, 41].

The generalized total rank conjecture concerns a bounded complex $P_\bullet$ of finite free modules with nonzero finite-length homology:

$$\sum_i \operatorname{rank}_R P_i \geq 2^d. \quad (6.4)$$

This is not equivalent to the module case. Iyengar and Walker produced counterexamples to the generalized assertion in odd characteristic [29]. Their examples do not settle the characteristic-two case.

Theorem 6.5 (VandeBogert, 2026 [39]). *Let R be a regular Noetherian domain of characteristic 2, and let P be a differential R -module admitting a finite projective flag. If $H(P) \neq 0$, then*

$$\operatorname{rank}_R(P) \geq 2^{\operatorname{codim}_R H(P)}, \quad \operatorname{codim}_R H(P) = \min_{\mathfrak{p} \in \operatorname{Supp}_R H(P)} \operatorname{ht} \mathfrak{p}. \quad (6.5)$$

In particular, (6.4) holds over regular local rings of characteristic 2.

7 Differential modules and the Tate construction

From now on, unless otherwise specified, A is a commutative ring in which $2 = 0$. The construction does not require a homological grading.

Definition 7.1. A *differential A -module* is a pair (P, d_P) consisting of an A -module P and an A -linear endomorphism $d_P : P \rightarrow P$ with $d_P^2 = 0$. Its homology is $H(P) = \ker d_P / \operatorname{im} d_P$. A *finite free flag* is a filtration

$$0 = P^{-1} \subseteq P^0 \subseteq \cdots \subseteq P^s = P$$

by direct summands with finite free subquotients and $d_P(P^j) \subseteq P^{j-1}$. A projective flag is defined similarly, with finite projective subquotients.

For two differential modules P, Q , the tensor product is equipped with $d_P \otimes 1 + 1 \otimes d_Q$. Its square is $2(d_P \otimes d_Q) = 0$. In the ungraded differential-module setting this formula uses the characteristic-two assumption; there are no Koszul signs attached to transposition.

Let $\tau : P \otimes_A P \rightarrow P \otimes_A P$ be the transposition $\tau(x \otimes y) = y \otimes x$, and put

$$\partial = d_P \otimes 1 + 1 \otimes d_P, \quad N = 1 + \tau.$$

The maps ∂ and N commute and satisfy $\partial^2 = N^2 = 0$.

Definition 7.2 (Tate-valued tensor square). The *Tate construction* of P is the differential A -module

$$\mathsf{T}_A(P) = (P \otimes_A P, D_P), \quad D_P = \partial + (1 + \tau). \quad (7.1)$$

For a morphism $f : P \rightarrow Q$, set $\mathsf{T}_A(f) = f \otimes f$.

The defining endomorphism is genuinely a differential:

$$D_P^2 = \partial^2 + \partial N + N \partial + N^2 = 0.$$

This is a differential module, not in general an ordinary $\mathbb{Z}$ -graded chain complex: the two summands of D_P need not have the same homological degree.

7.1 A two-dimensional calculation

Example 7.3. Let $V = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2$ and set $d_V = 0$. The basis of $V \otimes_{\mathbb{F}_2} V$ is

$$e_1 \otimes e_1, \quad e_1 \otimes e_2, \quad e_2 \otimes e_1, \quad e_2 \otimes e_2.$$

On these tensors $D_V = 1 + \tau$, so

$$\begin{aligned} \operatorname{im}(1 + \tau) &= \langle e_1 \otimes e_2 + e_2 \otimes e_1 \rangle, \\ \operatorname{ker}(1 + \tau) &= \langle e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2 \rangle. \end{aligned}$$

Thus

$$H(\mathsf{T}_{\mathbb{F}_2}(V)) = \langle [e_1 \otimes e_1], [e_2 \otimes e_2] \rangle \cong \mathbb{F}_2^2.$$

The off-diagonal orbit is acyclic, whereas each diagonal tensor contributes one homology class.

The calculation is not confined to the field $\mathbb{F}_2$. Its natural formulation records the Frobenius

twist.

Definition 7.4. For a ring A of characteristic two, let A_F denote A with its usual left A -action and right action $x \cdot a = xa^2$. For an A -module M , its *Frobenius pullback* is

$$F_A^*M = A_F \otimes_A M.$$

For a differential module (P, d_P) the pullback differential is $1 \otimes d_P$.

Lemma 7.5 (The diagonal calculation [39]). *If V is finite free over A with zero differential, then there is a natural isomorphism*

$$\gamma_V : F_A^*V \xrightarrow{\sim} H(\mathsf{T}_A(V)), \quad \gamma_V(1 \otimes v) = [v \otimes v].$$

Proof. The assignment $v \mapsto [v \otimes v]$ is additive in homology:

$$(v + w) \otimes (v + w) - v \otimes v - w \otimes w = (1 + \tau)(v \otimes w).$$

It is Frobenius-semilinear because $(av) \otimes (av) = a^2(v \otimes v)$. Hence it induces an A -linear map from $A_F \otimes_A V$. For a basis $e_1, \dots, e_s$ of V , the tensors $e_i \otimes e_j + e_j \otimes e_i$ with $i < j$ generate $\text{im}(1 + \tau)$, and the diagonal tensors represent a basis of $\ker(1 + \tau)/\text{im}(1 + \tau)$. The classes $[e_i \otimes e_i]$ are therefore a basis of the target, as $1 \otimes e_i$ are a basis of F_A^*V . $\square$

Remark 7.6. The naive map $v \mapsto v \otimes v$ is not additive in the tensor square. It becomes additive only after quotienting by the image of $1 + \tau$. It is therefore incorrect to replace lemma 7.5 by an asserted A -linear diagonal map $F_A^*V \rightarrow V \otimes_A V$.

7.2 Homotopy properties

There are two properties of the construction: it depends only on the homotopy class of a map and is additive on homotopy classes. These are proved, together with exactness, in [39, Lemma 2.3].

Proposition 7.7 (Homotopy and exactness [39]). *For finite projective differential A -modules, T_A preserves homotopies and is additive up to homotopy. Moreover, every short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

induces a long exact sequence in Tate homology, with the usual periodicity of differential-module homology.

Proof outline. For morphisms f, g , the difference between $(f + g) \otimes (f + g)$ and $f \otimes f + g \otimes g$ is $f \otimes g + g \otimes f$, which is null-homotopic for the Tate differential. The homotopy-invariance calculation is given explicitly in [39, Lemma 2.3]. For exactness, projectivity splits the underlying sequence. Filter $Y \otimes Y$ by

$$X \otimes X \subseteq (X \otimes Y) + (Y \otimes X) \subseteq Y \otimes Y.$$

The middle associated quotient is $(X \otimes Z) \oplus (Z \otimes X)$; transposition interchanges its two summands. If this is written as $M \oplus M$, the endomorphism $h(m_1, m_2) = (m_2, 0)$ satisfies $Dh + hD = 1$. The middle quotient is thus contractible. The two remaining associated pieces are $\mathsf{T}_A(X)$ and $\mathsf{T}_A(Z)$, which gives the claimed long exact homology sequence. $\square$

8 The Adams filtration and Frobenius homology

Suppose P admits a finite free flag and $H(P)$ has a finite free resolution

$$0 \longrightarrow F_s \xrightarrow{\delta_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \longrightarrow H(P) \longrightarrow 0. \quad (8.1)$$

A standard resolution-by-flags construction replaces P , up to homotopy equivalence, by a differential module X with a finite filtration

$$0 = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^s = X, \quad X^q/X^{q-1} \cong F_q,$$

whose induced adjacent differential is δ_q . The existence of this replacement is nontrivial; a construction and proof appear in [39, Lemma 3.1]. More precisely, that construction first supplies a quasi-isomorphism $X \rightarrow P$. Both X and P admit projective flags, so this quasi-isomorphism is a homotopy equivalence by [40, Lemma A.19]. This justifies replacing P by X when applying the homotopy-invariant Tate construction.

Applying Tate homology to the short exact sequences of the filtration forms a finite exact couple. Its spectral sequence satisfies

$$E_q^1 \cong H(\mathsf{T}_A(F_q)) \cong A_F \otimes_A F_q. \quad (8.2)$$

The last identification uses lemma 7.5. The differential $d^1 : E_q^1 \rightarrow E_{q-1}^1$ corresponds to $1 \otimes \delta_q$. Indeed, if $\tilde{v} \in X^q$ lifts $v \in F_q$, then, for $D = \partial + 1 + \tau$,

$$D(\tilde{v} \otimes \tilde{v} + d\tilde{v} \otimes \tilde{v}) = d\tilde{v} \otimes d\tilde{v}. \quad (8.3)$$

Under the diagonal identification the right side represents $1 \otimes \delta_q(v)$ at the next filtration level. Thus the second page is

$$\boxed{E_q^2 \cong \mathrm{Tor}_q^A(A_F, H(P))}. \quad (8.4)$$

Theorem 8.1 (Tate–Frobenius comparison [39]). *Suppose A is a commutative Noetherian ring of characteristic 2, A_F is flat as a right A -module, and P admits a finite free flag with $H(P)$ possessing a finite free resolution. Then*

$$H(\mathsf{T}_A(P)) \cong A_F \otimes_A H(P). \quad (8.5)$$

In particular, the conclusion holds when A is regular local of characteristic 2.

Proof. The filtration constructed above gives the finite spectral sequence (8.2)–(8.4), converging to $H(\mathsf{T}_A(P))$. Flatness of A_F gives $\mathrm{Tor}_q^A(A_F, H(P)) = 0$ for $q > 0$. Hence the second page is concentrated in filtration degree 0, where its term equals $A_F \otimes_A H(P)$. No further differential or extension can occur, so the sequence collapses and gives (8.5). For regular local A , flatness of Frobenius follows from Kunz’s theorem [32], and a finite free resolution exists by finite global dimension. $\square$

Remark 8.2. Since Frobenius is flat under the hypotheses of theorem 8.1,

$$A_F \otimes_A H(P) \cong H(F_A^* P).$$

One must distinguish the Frobenius pullback $F_A^* P$ from merely regarding P as a module with scalar action twisted by Frobenius.

9 Homology lengths and the rank bound

Assume $(R, \mathfrak{m}, k)$ is a regular local ring of characteristic two and dimension d . For a differential R -module C with finite-length homology set

$$h(C) = \ell_R H(C).$$

Lemma 9.1 (Tensor-product estimate [39]). *If X has finite-length homology and Y admits a finite free flag, then*

$$h(X \otimes_R Y) \leq \mathrm{rank}_R(Y) h(X).$$

Proof. Let the successive quotients of the flag of Y have ranks $r_0, \dots, r_s$. Tensoring with X produces a split filtration of $X \otimes_R Y$, with subquotients isomorphic to $X^{\oplus r_j}$. The long exact homology sequences bound the homology length of the whole by the sum of those of the successive quotients. Thus

$$h(X \otimes_R Y) \leq \sum_{j=0}^s r_j h(X) = \mathrm{rank}_R(Y) h(X).$$

□

Lemma 9.2 (Frobenius length [32, 39]). *For every finite-length R -module N ,*

$$\ell_R(F_R^* N) = 2^d \ell_R(N). \quad (9.1)$$

Proof. By Kunz's theorem, F_R^* is exact, so it suffices to prove the formula for $N = k$. If $x_1, \dots, x_d$ is a regular system of parameters, then

$$F_R^* k \cong R/(x_1^2, \dots, x_d^2),$$

which has length 2^d because the parameters form a regular sequence. A composition series proves the general formula. □

The remaining comparison is a deformation statement. It is useful to separate it from the immediate filtration estimate: there is no elementary monotonicity principle for homology under an arbitrary change of differential.

Proposition 9.3 (Deformation comparison [39]). *Let P be a differential R -module admitting a finite free flag and suppose $0 < h(P) < \infty$. In the regular local setting above,*

$$h(\mathbb{T}_R(P)) \leq h(P \otimes_R P). \quad (9.2)$$

Explanation of the length argument. Set $B = R[s]_{(\mathfrak{m}, s)}$ and $C = ((P \otimes_R P) \otimes_R B, \partial + s(1 + \tau))$. This interpolates between ∂ at $s = 0$ and the Tate differential after passing to the prime $\mathfrak{p} = \mathfrak{m}B$ and rescaling by the unit s . Explicitly, write $P_{\mathfrak{p}} = P \otimes_R B_{\mathfrak{p}}$ and set

$$Q = (P_{\mathfrak{p}}, s^{-1} d_{P_{\mathfrak{p}}}).$$

Then, on $P_{\mathfrak{p}} \otimes_{B_{\mathfrak{p}}} P_{\mathfrak{p}}$,

$$\partial_{P_{\mathfrak{p}}} + s(1 + \tau) = s(\partial_Q + (1 + \tau)).$$

Multiplying a differential by a unit preserves its kernel and image. Thus $H(C_{\mathfrak{p}}) \cong H(\mathbb{T}_{B_{\mathfrak{p}}}(Q))$; also $H(Q) \cong H(P_{\mathfrak{p}})$. The free flag is preserved by this rescaling, so the Tate–Frobenius comparison applies to Q . Let $M = H(C)$. The homology sequence associated to multiplication by s shows that M/sM embeds in $H(P \otimes_R P)$, hence $\ell_B(M/sM) \leq h(P \otimes_R P)$. The multiplicity estimate for this at-most-one-dimensional B -module gives

$$\ell_{B_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq e(s; M) \leq \ell_B(M/sM).$$

Over the regular local ring $B_{\mathfrak{p}}$ the Tate–Frobenius comparison identifies this generic-fiber length with $2^d h(P)$. Here $R \rightarrow B_{\mathfrak{p}}$ is flat with maximal ideal $\mathfrak{m}B_{\mathfrak{p}}$ and residue field $k(s)$, so a composition series gives $\ell_{B_{\mathfrak{p}}} H(P_{\mathfrak{p}}) = \ell_R H(P) = h(P)$. Comparing it with $h(\mathbb{T}_R(P)) = 2^d h(P)$

yields (9.2). The multiplicity step and the generic-fiber identification are the substantive parts of this argument; see [39, Proposition 2.5 and Theorem 2.8]. $\square$

Theorem 9.4 (Local total-rank inequality [39]). *Let $(R, \mathfrak{m}, k)$ be regular local of dimension d and characteristic 2. If P admits a finite free flag and $0 < h(P) < \infty$, then*

$$\mathrm{rank}_R(P) \geq 2^d.$$

Proof. Apply theorem 8.1 and lemma 9.2 to obtain $h(\mathrm{T}_R(P)) = 2^d h(P)$. By proposition 9.3 and lemma 9.1,

$$\underbrace{2^d h(P)}_{h(\mathrm{T}_R(P))} \leq h(P \otimes_R P) \leq \mathrm{rank}_R(P) h(P). \quad (9.3)$$

As $h(P) > 0$, canceling gives the assertion. $\square$

Proof of theorem 6.5. Set $N = H(P)$ and choose a prime $\mathfrak{p}$ of minimal height in $\mathrm{Supp}_R N$. Then $R_{\mathfrak{p}}$ is regular local, $N_{\mathfrak{p}} \neq 0$ has finite length, and $P_{\mathfrak{p}}$ admits a finite free flag: finite projective modules over local rings are free. Therefore theorem 9.4 gives

$$\mathrm{rank}_{R_{\mathfrak{p}}} P_{\mathfrak{p}} \geq 2^{\dim R_{\mathfrak{p}}} = 2^{\mathrm{codim}_R N}.$$

As R is a domain, the rank of a finite projective module is constant across $\mathrm{Spec} R$; hence its localization has rank $\mathrm{rank}_R P$, proving (6.5). $\square$

10 Consequences for group actions

Now we can connect the characteristic-two rank theorem back to the topological problem with which it began.

Corollary 10.1 (Carlsson’s conjecture in characteristic two [39]). *Let $(\mathbb{Z}/2\mathbb{Z})^r$ act freely and continuously on a nonempty finite CW complex X . Then*

$$\sum_i \dim_{\mathbb{F}_2} H_i(X; \mathbb{F}_2) \geq 2^r.$$

Proof. Carlsson’s Koszul-dual algebraic model associates to the equivariant problem a finite free differential graded module over $\mathbb{F}_2[x_1, \dots, x_r]$ whose total free rank is the total $\mathbb{F}_2$ -homology dimension in the assertion. The relevant homology has finite length and is nonzero for a nonempty free finite complex. The free flag and regularity hypotheses required for theorem 6.5 are established in the algebraic construction; see [39, Corollary B and Lemma 2.10]. Its rank inequality yields the desired estimate. $\square$

Corollary 10.2 (Sphere-rank inequality). *If $(\mathbb{Z}/2\mathbb{Z})^r$ acts freely on a finite CW complex homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_m}$ with $n_j \geq 1$, then $r \leq m$.*

Proof. By the Künneth formula the total mod-two homology dimension is 2^m . Apply corollary 10.1 to obtain $2^r \leq 2^m$. $\square$

Remark 10.3. The more general theorem in [39] also bounds the homology of finite CW complexes with nonfree actions in terms of the largest stabilizer rank.

Tor Algebra Classification for Rings from General Points

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KUMUNU 2026 ■ Sunday, October 11, 2026

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Abstract

These notes record the construction and classification of Tor algebras associated with general points of projective three-space. We first recall the multiplicative structure on $\mathrm{Tor}_*^Q(Q/I, k)$ and the numerical invariants distinguishing the classes in embedding codepth three and, in the Gorenstein case, codepth four. A finite set of points $X \subset \mathbb{P}^3$ gives both a one-dimensional coordinate ring $Q/I(X)$ and an Artinian Gorenstein apolar ring $Q/\mathrm{Ann}(F_X)$, where F_X is a sum of powers of linear forms. We explain the classification by the number of points and reconstruct the Betti tables displayed for six points. The photographed lecture is supplemented, with attribution, by the precise hypotheses and results of Fagerstrom's accompanying preprint.

11 Tor algebras and low-codepth classification

Let $Q = k[x_0, \dots, x_N]$ be a standard graded polynomial ring over a field k , and let $R = Q/I$ with $I \subseteq (x_0, \dots, x_N)^2$. Write

$$\mathbf{F}: 0 \longrightarrow F_c \xrightarrow{d_c} F_{c-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow R \longrightarrow 0$$

for the minimal graded free resolution of R over Q . Auslander–Buchsbaum gives $c = \dim Q - \mathrm{depth} R$; when R is Cohen–Macaulay, this is also $\mathrm{codim} R$. Its length c is commonly called the *embedding codepth* when Q is a minimal regular presentation of R .

Definition 11.1 (Tor algebra). The graded k -algebra associated with R is

$$A = A(R) = \bigoplus_{i=0}^c \mathrm{Tor}_i^Q(R, k) \cong H_*(\mathbf{F} \otimes_Q k).$$

It carries its homological grading A_i and, in the polynomial case, an additional internal grading

$$[A_i]_j = \mathrm{Tor}_i^Q(R, k)_j, \quad \beta_{i,j}^Q(R) = \dim_k [A_i]_j.$$

The product is bigraded: $[A_i]_a [A_j]_b \subseteq [A_{i+j}]_{a+b}$.

For codepth at most three the minimal free resolution admits a differential graded algebra structure. Its product induces the Tor-algebra structure, whose classification is independent of choices. This is the point of the classical results of Buchsbaum–Eisenbud, Weyman, and Avramov–Kustin–Miller [8, 4, 5]. In codimension four there is a corresponding classification for Gorenstein quotients due to Kustin and Miller [30]. The classifications describe products, not merely vector-space dimensions.

11.1 The invariants in codepth three

Assume $c = 3$ and put

$$p = \dim_k(A_1 A_1), \quad q = \dim_k(A_1 A_2), \quad r = \text{rank}_k(A_2 \longrightarrow \text{Hom}_k(A_1, A_3)),$$

where the map sends y to $(x \mapsto xy)$. The following table reproduces the numerical classification recorded on the board; its realization comes from the classification theorem [4, 5].

Table 1. Classes in embedding codepth three.

Class	p	q	r
$C(3)$	3	1	3
T	3	0	0
B	1	1	2
$G(r)$	0	1	r
$H(p, q)$	p	q	q

Remark 11.2. The numerical triple (p, q, r) does not always determine the class: T and $H(3, 0)$ both have $(p, q, r) = (3, 0, 0)$. Additional structural information is needed in that case. In particular, if R is a Cohen–Macaulay almost complete intersection of codepth three and type $n = \dim_k A_3$, then $n = 2$ gives $H(3, 2)$, odd $n \geq 3$ gives T , and even $n \geq 4$ gives $H(3, 0)$ [5, 23].

In class $H(0, 0)$ the product of any two elements of positive homological degree is zero. In the setting of these codepth-three quotients this is equivalent to the familiar Golod case $H(0, 0) \iff \text{Golod}$.

11.2 Gorenstein quotients in codepth four

For a codimension-four Gorenstein ring let

$$\bar{p} = \dim_k(A_1 A_1), \quad \bar{q} = \dim_k(A_1 A_2).$$

Poincaré duality makes the multiplication nondegenerate in complementary degrees and gives the following numerical possibilities, in the notation used by the `TorAlgebra` package and in [23, 30].

Table 2. Gorenstein Tor-algebra classes in codepth four.

Class	$\bar{p}$	$\bar{q}$
$C(4)$	6	4
GT	3	3
GS	0	0
$GH(\bar{p}), \bar{p} \geq 1$	$\bar{p}$	$\bar{p} + 1$

Remark 11.3. The label GS refers to the case of trivial products A_1A_1 and A_1A_2 , not to a trivial Tor algebra. A Gorenstein algebra of codepth four still has its nondegenerate top-degree pairings, for instance $A_1A_3 \rightarrow A_4$ and $A_2A_2 \rightarrow A_4$. Thus the two zero entries in [table 2](#) must not be interpreted as saying that every product of positive-degree classes vanishes.

12 From points to apolar Gorenstein rings

From now on take $Q = k[x_0, x_1, x_2, x_3]$, with k algebraically closed of characteristic zero, as in [23]. Let

$$X = \{P_1, \dots, P_t\} \subseteq \mathbb{P}_k^3, \quad P_i = [a_{i0} : a_{i1} : a_{i2} : a_{i3}].$$

12.1 The coordinate ring

The homogeneous vanishing ideal of X is

$$J = I(X) = \bigcap_{i=1}^t I(P_i), \quad R_X = Q/J.$$

For a nonempty reduced finite set of projective points, R_X is a one-dimensional Cohen–Macaulay graded ring. Accordingly,

$$\dim R_X = 1, \quad \operatorname{codim}_Q R_X = 3.$$

For general points, the relevant open condition guarantees the generic Hilbert function and the expected minimal resolution. In particular,

$$\dim_k (R_X)_d = \min \left\{ \binom{d+3}{3}, t \right\}.$$

12.2 Apolar construction

Introduce another polynomial ring $D = k[y_0, y_1, y_2, y_3]$. We let Q act on D by constant-coefficient differential operators:

$$x_i \circ F = \frac{\partial F}{\partial y_i}, \quad f \circ F = f\left(\frac{\partial}{\partial y_0}, \dots, \frac{\partial}{\partial y_3}\right) F.$$

Associate to P_i the linear form

$$\ell_i = a_{i0}y_0 + \dots + a_{i3}y_3$$

and choose an integer $s > 0$. Let

$$F_X = \ell_1^s + \dots + \ell_t^s \in D_s.$$

One then forms the annihilator ideal and its quotient,

$$I = \text{Ann}_Q(F_X) = \{f \in Q : f \circ F_X = 0\}, \quad S_X = Q/I. \quad (12.1)$$

By the Macaulay inverse-system correspondence, S_X is an Artinian Gorenstein graded algebra with socle degree s , provided $F_X \neq 0$; this nonvanishing holds for the general configurations and degrees under consideration. Thus $\text{codim}_Q S_X = 4$.

The two constructions can be summarized by the diagram

$$\begin{array}{ccc} X = \{P_1, \dots, P_t\} & \xrightarrow{\text{choose representatives}} & F_X = \sum_i \ell_i^s \\ \text{coordinate ring} \downarrow & & \downarrow \text{apolar ring} \\ Q/I(X) & & Q/\text{Ann}_Q(F_X). \end{array}$$

The arrow across the top involves the choice of representatives for the projective points. The two quotients are not the same: J is the vanishing ideal of X , whereas I is the annihilator of a homogeneous form under differentiation.

Remark 12.1 (Waring rank versus number of summands). The identity $F_X = \sum_{i=1}^t \ell_i^s$ expresses F_X as a sum of t powers. Its Waring rank is *at most* t ; equality requires a further genericity condition.

13 The Betti table of six general points

Here we use the Macaulay2 convention that a row labeled $j - i$ contains the entry $\beta_{i,j}$.

Example 13.1 (Six points in $\mathbb{P}^3$). For six general points, the Hilbert function and Hilbert series are

$$(H_{R_X}(d))_{d \geq 0} = (1, 4, 6, 6, \dots), \quad \text{Hilb}(R_X; z) = \frac{1 + 3z + 2z^2}{1 - z}.$$

The generic Betti numbers established by Ballico–Geramita [9] give the minimal Q -resolution

$$0 \longrightarrow Q(-5)^2 \longrightarrow Q(-3)^2 \oplus Q(-4)^3 \longrightarrow Q(-2)^4 \longrightarrow Q \longrightarrow R_X \longrightarrow 0. \quad (13.1)$$

Equivalently, the photographed table reads

	0	1	2	3
total	1	4	5	2
0	1	—	—	—
1	—	4	2	—
2	—	—	3	2

As a numerical check, the numerator of the Hilbert series computed from equation (13.1) is

$$1 - 4z^2 + 2z^3 + 3z^4 - 2z^5 = (1 - z)^3(1 + 3z + 2z^2).$$

Hence the resolution is compatible with the Hilbert function above. This check does not on its own prove the claimed exactness; that follows from the generic-resolution result cited above.

The coordinate ring in Example 13.1 has four minimal quadratic generators, hence it is a codimension-three almost complete intersection. Its Cohen–Macaulay type is $\dim_k A_3 = 2$. By the codepth-three classification, the associated Tor algebra is of class $H(3, 2)$. In particular,

$$\dim_k(A_1 A_1) = 3, \quad \dim_k(A_1 A_2) = 2.$$

The dimensions are multiplication invariants and are not read merely from the four total Betti numbers.

14 Doubling and the reflected Betti table

Definition 14.1 (Doubling). Let R be a one-dimensional Cohen–Macaulay graded ring with canonical module ω_R . A quotient S is a doubling of R if, for a suitable integer

$a > 0$, there is an exact sequence

$$0 \longrightarrow \omega_R(-a) \longrightarrow R \longrightarrow S \longrightarrow 0. \quad (14.1)$$

The construction under the usual hypotheses produces a Gorenstein ring.

Write $r = \text{reg}(R_X)$. A result of Boij, in the form used by Fagerstrom, gives that for $s \geq 2r - 1$, the apolar algebra S_X is a doubling of R_X . Under the stronger assumption $s \geq 2r$, their graded Betti numbers are related by

$$\beta_{i,j}^Q(S_X) = \beta_{i,j}^Q(R_X) + \beta_{4-i,s+4-j}^Q(R_X). \quad (14.2)$$

The second term is the reflected contribution of the canonical module. The index $4 - i$ is dictated by the four variables in Q ; it is not the codepth-three index $3 - i$.

For the six-point example, $r = 2$. From [equation \(14.2\)](#), for $s \geq 4$ and in particular throughout the main theorem's allowed high-degree range, we obtain

$$\begin{aligned} 0 &\longrightarrow Q(-s-4) \\ &\longrightarrow Q(-5)^2 \oplus Q(-s-2)^4 \\ &\longrightarrow Q(-3)^2 \oplus Q(-4)^3 \oplus Q(-s)^3 \oplus Q(-s-1)^2 \\ &\longrightarrow Q(-2)^4 \oplus Q(-s+1)^2 \longrightarrow Q \longrightarrow S_X \longrightarrow 0. \end{aligned} \quad (14.3)$$

Here $Q(-s+1)$ means $Q(-(s-1))$. The ranks are therefore

$$(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)(S_X) = (1, 6, 10, 6, 1),$$

consistent with the Gorenstein symmetry.

When $s \geq 7$, the two pieces of the Betti table occupy distinct rows, precisely as pictured on the board:

	0	1	2	3	4
total	1	6	10	6	1
0	1	—	—	—	—
1	—	4	2	—	—
2	—	—	3	2	—
$s-2$	—	2	3	—	—
$s-1$	—	—	2	4	—
s	—	—	—	—	1

The upper two rows coincide with the Betti table of R_X . The reflected rows arise from the shifted canonical module, while the final entry 1 records the type of the Artinian Gorenstein ring.

Remark 14.2 (The degree restriction). For the six-point example, where this regularity is 2, the bound $s \geq 7$ agrees with the useful high-degree range. For a general number of points, however, the precise theorem in [23, Theorem 4.14] is formulated using $\mu(t)$, not merely the regularity $d(t)$ of the coordinate ring. We use the published hypotheses in [theorem 15.1](#) rather than silently extrapolating an approximate board annotation.

15 Classification by the number of general points

Put

$$\mu(t) = \min \left\{ j \geq 1 : \binom{j+3}{3} > t \right\}.$$

The notation $\mu(t)$ records the first degree in which a general t -point configuration has a nonzero defining equation. It is distinct from the regularity $d(t) = \min \{ j : \binom{j+3}{3} \geq t \}$ in the boundary cases where the binomial coefficient equals t .

Theorem 15.1 (Classification for general points; Fagerstrom). *Assume k is algebraically closed of characteristic zero and $t \geq 4$. Let $X \subset \mathbb{P}_k^3$ consist of t general points, let $R_X = Q/I(X)$, and let $S_X = Q/\text{Ann}_Q(F_X)$ arise from a general sum of t sth powers of linear forms, with the appropriate genericity conditions on its Betti numbers and Waring rank. If*

$$2\mu(t) \leq s \leq 3\mu(t) - 4 \quad \text{or} \quad s \geq 3\mu(t) + 1, \quad (15.1)$$

then the Tor-algebra classes are

t	4	5	6	7	8	≥ 9
$\text{Tor}_*^Q(R_X, k)$	$H(0, 0)$	$G(5)$	$H(3, 2)$	T	$H(1, 0)$	$H(0, 0)$
$\text{Tor}_*^Q(S_X, k)$	GS	$GH(5)$	$GH(5)$	GT	$GH(1)$	GS

In each case the Tor algebra has maximum multiplication rank in the sense specified in [23, Definition 4.2].

Remark 15.2. The range $2\mu(t) \leq s \leq 3\mu(t) - 4$ may be empty for small $\mu(t)$. For $4 \leq t \leq 9$, one has $\mu(t) = 2$, so the displayed classification of apolar algebras applies in particular for $s \geq 7$. The generality conditions in the theorem cannot be dropped: special point sets and arbitrary apolar forms can have different Tor products.

15.1 Why the codepth-three entries take these values

The preprint supplies the required proofs [23, Theorem 4.6]. The structure can nevertheless be seen from a small amount of numerical information. Four general points give the class $H(0, 0)$. Five general points have a codimension-three Gorenstein coordinate ring whose

minimal resolution has five generators; thus its class is $G(5)$. For six and seven points the defining ideal has four minimal generators. The Cohen–Macaulay types are respectively 2 and 3, and the almost-complete-intersection classification gives $H(3, 2)$ and T . For eight points, the two quadratic minimal generators yield a nonzero Koszul product in Tor . Combined with degree bounds this forces $H(1, 0)$. For nine or more general points, grading restrictions force the multiplication invariants p and q to vanish, giving $H(0, 0)$.

15.2 The codepth-four passage

The apolar rings are Artinian Gorenstein of codepth four. Their Tor algebras consequently satisfy Poincaré duality, and the multiplication in low homological degrees is sharply constrained. In the degree range [equation \(15.1\)](#), the doubling construction transfers multiplication information from the coordinate ring. The key numerical bounds for the case $\mu(t) = 2$ take the form

$$\begin{aligned}\bar{p} &\leq \min\left\{\binom{\alpha_1}{2}, \beta_2\right\} + \min\{\alpha_1\beta_3, \alpha_2\}, \\ \bar{q} &\leq \min\{\alpha_1\alpha_2, \beta_3\} + \alpha_1,\end{aligned}$$

where $\alpha_i = \beta_{i,i+1}^Q(R_X)$ and $\beta_i = \beta_{i,i+2}^Q(R_X)$. These are bounds arising from the internal grading; their attainment requires additional arguments.

For six general points we have $(\alpha_1, \alpha_2, \alpha_3) = (4, 2, 0)$ and $(\beta_1, \beta_2, \beta_3) = (0, 3, 2)$. Consequently

$$\bar{p} \leq \min\{6, 3\} + \min\{8, 2\} = 5, \quad \bar{q} \leq \min\{8, 2\} + 4 = 6.$$

The structural theorem for a codimension-four Gorenstein ideal with six minimal generators identifies the class as $GH(5)$, whose invariants are $(\bar{p}, \bar{q}) = (5, 6)$. In this example the maximum permissible multiplication ranks are achieved. This calculation explains the six-point entry in the last row of [theorem 15.1](#).

For $t = 7$ the same technique, together with the classification of codepth-four Gorenstein structures, identifies GT rather than $GH(3)$. For $t = 8$ it yields $GH(1)$. The remaining cases have $\bar{p} = \bar{q} = 0$, giving GS . Details of the transfer and the classification in each case are in [\[23, §4\]](#).

16 What the classification does and does not assert

The calculation of graded Betti numbers gives the dimensions of the bigraded components $[A_i]_j$. It does not, by itself, prescribe the multiplication maps

$$[A_i]_a \otimes_k [A_j]_b \longrightarrow [A_{i+j}]_{a+b}.$$

This is why the classification of the Tor algebra contains information beyond a minimal free resolution viewed solely as a complex of free modules. The generality assumption ensures

that particular multiplicative ranks attain the appropriate maximal bounds, and those ranks distinguish the classes in the tables above.

There is a second point of care. For a fixed special set X of points, the coordinates of X can affect multiplication without changing all its numerical Betti invariants. Indeed, [23, §5] provides examples of point configurations with the same total Betti numbers but different Tor-algebra classes. The theorem is therefore a classification for the specified general families, not a classification of arbitrary finite point sets in projective space.

17 Other invited and graduate-student talks

17.1 Reconstruction Results for Projective Varieties

Prashanth Sridhar

University of Alabama ■ Saturday, October 10, 2:00–3:00 p.m.

Abstract The problem of reconstructing a projective variety from categorical data extends well beyond the classical theory of derived categories with an ample canonical or anticanonical bundle. The talk describes constructions of dg-categories and n -stable categories arising naturally from projective varieties, including formulations built from sheaves of differential graded algebras. These enhanced categories retain enough geometric information to recover the original varieties under hypotheses that go beyond the familiar Bondal–Orlov and Ballard reconstruction settings. The focus is on the reconstruction mechanism and the additional flexibility afforded by differential graded enhancements.

References.

- Michael K. Brown and Prashanth Sridhar, *Orlov’s Theorem for dg-algebras*, [arXiv:2312.17422](#).
- A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*;
- M. Ballard, *Derived categories of sheaves on singular schemes with an application to reconstruction*.

17.2 Quasi-Gorenstein Morphisms of DG-rings and Exact Sequences

Zachary Nason

University of Nebraska–Lincoln ■ Saturday, October 10, 3:20–3:50 p.m.

Abstract Quasi-Gorenstein local homomorphisms provide a relative interpretation of the Gorenstein property: for instance, the canonical map from a local ring to its residue field is quasi-Gorenstein precisely when the source ring is Gorenstein. The lecture moves this notion to commutative local DG-rings and examines how its homological behavior can recognize exact sequences. In particular, exactness for a sequence of elements of a Noetherian local ring is characterized through quasi-Gorenstein morphisms between associated Koszul DG-algebras. This extends the framework of Avramov–Henriques–Şega. The work is joint with Andrew J. Soto Levins and Ryan Watson.

References

- Z. Nason, A. J. Soto Levins and R. Watson, *Quasi-Gorenstein morphisms of commutative local dg-algebras*, arXiv:2605.02029.

17.3 Test Ideal Can Be Computed by a Boundary

Rahul Ajit

University of Kansas ■ Saturday, October 10, 3:50–4:50 p.m.

Abstract Suppose that R is a normal, reduced, F -finite ring of characteristic $p > 0$. The talk announces that the big test ideal of R can be obtained from an effective boundary: there exists an effective $\mathbb{Q}$ -divisor Δ on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$ -Cartier of index prime to p and

$$\tau_b(R) = \tau_b(R, \Delta).$$

Thus the test ideal of a ring without an assumed $\mathbb{Q}$ -Gorenstein condition can be realized by a pair admitting better Cartier-index control. The assertion addresses a question attributed in the conference abstract to Schwede, Smith, and others; related questions may also be discussed.

References

- R. Ajit and H. Simper, *Test Modules of Extended Rees Algebras*, arXiv:2509.01693. This is related work, not a proof reference for the boundary-existence statement reported above from the conference abstract.

17.4 Local Cohomology in t -Stanley-Reisner Rings

Olivia Strahan

University of New Mexico ■ Saturday, October 10, 5:10–6:10 p.m.

Abstract synopsis. Classical Stanley–Reisner theory relates the cohomology of a simplicial complex to local cohomology and other invariants of its face ring. This lecture develops a mixed-characteristic counterpart. Beginning with a discrete valuation ring (V, t) whose fraction field has characteristic zero and whose residue field has characteristic $p > 0$, one lets the uniformizer t replace one of the variables in a squarefree monomial construction. The resulting t -Stanley–Reisner rings correspond to simplicial complexes equipped with a distinguished vertex. The appropriate analogue of Hochster’s formula is no longer expressed solely through simplicial cohomology with constant coefficients: a cellular sheaf, with coefficients varying according to the faces, gives the correct local-cohomological description. The work is joint with Mel Hochster.

References:

- M. Hochster and O. Strahan, *Stanley-Reisner Theory in Mixed Characteristic*, [arXiv:2609.38947](#).

17.5 Frobenius Stability as Measure of Singularity

Ryan Hunter

University of Kansas ■ Sunday, October 11, 10:00–10:30 a.m.

Abstract: In positive characteristic, Frobenius actions on local cohomology modules can record subtle features of singularities. The talk studies Frobenius stability for such modules and relates it to the way associated local cohomology modules behave under rings of differential operators. The comparison is made concrete for selected familiar singular hypersurfaces, providing a route from Frobenius-invariant submodule structure to differential-operator-theoretic information. The description is a programmatic statement of the lecture’s subject, rather than a claim that every possible converse or characterization was proved.

References:

- F. Enescu, *Local cohomology and F -stability*, [arXiv:0904.4425](#);
- M. Blickle, *The intersection homology D -module in finite characteristic*, [arXiv:math/0110244](#).

17.6 Multiplicity of Graded Families of $\mathfrak{m}$ -primary Ideals

Dale Cutkosky

University of Missouri ■ Sunday, October 11, 11:30 a.m.–12:30 p.m.

Abstract: The Hilbert–Samuel multiplicity of an $\mathfrak{m}$ -primary ideal extends naturally to graded families of $\mathfrak{m}$ -primary ideals, but the direct limit of normalized colengths need not behave uniformly over arbitrary Noetherian local rings. The lecture introduces a multiplicity defined for these general graded families in a way that always yields an appropriate limit. It recovers the usual multiplicity for adic families and agrees with asymptotic volume under the standard hypotheses ensuring that volume is defined. The resulting theory admits counterparts of mixed-multiplicity constructions, Rees-type results, and the Minkowski inequality and its equality case. Its proofs use intersection products associated to blowups, with a particularly geometric ingredient for Minkowski equality.

References:

- S. D. Cutkosky, *Multiplicities of graded families of ideals on Noetherian local rings*, [arXiv:2603.06844](https://arxiv.org/abs/2603.06844).

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