25th Anniversary

Annual conference for commutative algebraists in the Great Plains region

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Conference Notes

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Celebrating 25 years of a regional conference in commutative algebra

Named for its original participating institutions—the University of Kansas, the University of Missouri–Columbia, and the University of Nebraska–Lincoln—the conference has been held annually since 1999. It provides a setting to share cutting-edge mathematical research and promote collaborations among algebraists in and near the Great Plains region.

The conference now attracts participants from a broader geographical region, promotes connections between commutative algebra and related areas, and showcases the research of early career mathematicians.

Contents

1 Linkage and Licci Ideals by Craig Huenke

1.1 Notation and Setup

Throughout, let either $(R, \mathfrak{m})$ be a regular local ring, or let

$$R = K[x_1, \ldots, x_n]$$

be a polynomial ring over a field K with homogeneous maximal ideal

$$\mathfrak{m} = (x_1, \ldots, x_n).$$

Let $I \subseteq R$ be a proper ideal. In the graded case, we consider the minimal graded free resolution of R/I:

$$\cdots \longrightarrow \bigoplus_j R(-d_{cj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(-d_{1j}) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Here $c = \operatorname{codim} I$. For the i-th term in the resolution, define

$$D_i = \max_j d_{ij}, \qquad d_i = \min_j d_{ij}.$$

Thus D_i and d_i record, respectively, the largest and smallest degree shifts occurring in homological degree i. Let

$$\alpha_1, \alpha_2, \ldots, \alpha_c \in I$$

be a regular sequence. We write

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_c)$$

for the complete intersection ideal generated by this sequence.

Remark 1.1. Since R is regular local, or standard graded polynomial, the usual homological invariants of R/I are well behaved. If R/I is Cohen–Macaulay of codimension c, then the projective dimension of R/I is c by the Auslander–Buchsbaum formula in the local case and its graded analogue in the polynomial case. This is the setting in which linkage through complete intersections is most naturally formulated.

1.2 Linkage

Definition 1.2 (Linkage)**.** Let $I, J \subsetneq R$ be proper ideals of the same codimension c. We say that I and J are *linked* by the regular sequence

$$\underline{\alpha} = \alpha_1, \ldots, \alpha_c \subseteq I \cap J$$

if
$$J = (\underline{\alpha}) : I \qquad \text{and} \qquad I = (\underline{\alpha}) : J.$$
In this case we write
$$I \sim_{\underline{\alpha}} J.$$

Definition 1.3 (Peskine–Szpiro)**.** Two proper ideals I and J are linked via a regular sequence $\underline{\alpha}$ if
$$(\underline{\alpha}) : I = J \qquad \text{and} \qquad (\underline{\alpha}) : J = I.$$

Remark 1.4. The two preceding definitions express the same notion. The Peskine–Szpiro formulation emphasizes the symmetry of the colon operation with respect to the complete intersection $(\underline{\alpha})$.

Definition 1.5 (Linkage class)**.** Two ideals I_0 and I_ℓ are in the same *linkage class* if there exists a sequence of linkages
$$I_0 \sim_{\underline{\alpha}_0} I_1 \sim_{\underline{\alpha}_1} I_2 \sim_{\underline{\alpha}_2} \cdots \sim_{\underline{\alpha}_{\ell-1}} I_\ell.$$

Remark 1.6. When one distinguishes between an odd number and an even number of links, one obtains the notions of *liaison class* and *even liaison class*. In projective geometry, even liaison is especially important because several cohomological invariants are preserved up to shift under even liaison.

1.3 Linkage via Free Resolutions

Suppose I is linked to J through the complete intersection
$$(\underline{\alpha}) = (\alpha_1, \ldots, \alpha_c) \subseteq I.$$
Then
$$J = (\underline{\alpha}) : I.$$
A comparison map from a free resolution of R/I to the Koszul complex on $\underline{\alpha}$ yields, by a mapping cone construction, a free resolution of R/J. More explicitly, if
$$F_\bullet : \quad 0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is a free resolution of R/I, and

$$K_\bullet(\underline{\alpha}): \quad 0 \longrightarrow K_c \longrightarrow K_{c-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow R \longrightarrow R/(\underline{\alpha}) \longrightarrow 0$$

is the Koszul resolution of $R/(\underline{\alpha})$, then the inclusion $(\underline{\alpha}) \subseteq I$ induces a comparison morphism

$$K_\bullet(\underline{\alpha}) \longrightarrow F_\bullet.$$

Dualizing and taking the appropriate mapping cone produces a resolution of the linked ideal. The lecture notes record the following schematic diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_c \arrow[r] \arrow[d, "a"] & \cdots \arrow[r] & R \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] & \cdots \arrow[r] & R^c \arrow[r] & R/(\underline{\alpha}) \arrow[r] & 0.
\end{tikzcd}$$

The notes further indicate the formula

$$J = (\underline{\alpha}) : I = I_1(a) + (\alpha_1, \ldots, \alpha_c),$$

where $I_1(a)$ denotes the ideal generated by the entries of the comparison matrix a appearing in the leftmost vertical map.

Remark 1.7. The above formula is a standard phenomenon in linkage. If the top map in the comparison between the free resolution of R/I and the Koszul complex is represented by a matrix a, then the entries of a, together with the complete intersection generators $\alpha_1, \ldots, \alpha_c$, generate the residual ideal $J = (\underline{\alpha}) : I$ in many common presentations. The precise statement depends on the chosen resolutions and comparison maps.

Remark 1.8 (Alexander duality). In the special setting of squarefree monomial ideals, Alexander duality is a powerful combinatorial duality that relates generators and faces of complementary simplicial complexes. Although linkage and Alexander duality can interact in special monomial situations, one should not identify the linked ideal $(\underline{\alpha}) : I$ with an Alexander dual without additional hypotheses.

Example 1.9. Consider an ideal of the form

$$(x_1^2, \ldots, x_n^2, J),$$

where J is a squarefree monomial ideal. Taking Frobenius powers, written symbolically as $x^{[q]}$, produces related families of monomial ideals. In positive characteristic, Frobenius powers preserve much of the monomial structure and can be used to study linkage-like constructions for such ideals.

1.4 Hartshorne–Rao Modules

Remark 1.10 (Hartshorne–Rao). Let $C \subseteq \mathbb{P}^3$ be a locally Cohen–Macaulay curve. Its Hartshorne–Rao module is

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t)).$$

This module is, up to degree shift, an invariant of the even liaison class of C.

Remark 1.11. The Hartshorne–Rao module measures the failure of the curve C to be arithmetically Cohen–Macaulay. In particular, C is arithmetically Cohen–Macaulay if and only if $M(C) = 0$. Since even liaison preserves $M(C)$ up to shift, the Hartshorne–Rao module is one of the central invariants in the liaison theory of space curves.

1.5 Licci Ideals

Definition 1.12 (Licci ideal)**.** An ideal I is called *licci* if it lies in the linkage class of a complete intersection. Equivalently, I is licci if there exists a finite sequence of links

$$I = I_0 \sim I_1 \sim \cdots \sim I_s$$

such that I_s is generated by a regular sequence.

Remark 1.13. The word "licci" is an abbreviation for "linked to a complete intersection."

1.5.1 Necessary Conditions for Being Licci

Several necessary conditions are known for an ideal to be licci.

Theorem 1.14 (Buchweitz)**.** *If I is licci, then I is strongly unobstructed.*

Remark 1.15. The lecture notes record the condition in the form

$$J/I^2 \otimes \omega_{R/I} \quad \text{is Cohen–Macaulay over } R/I.$$

Theorem 1.16 (Huneke). *If I is licci, then I is strongly Cohen–Macaulay. In particular, the Koszul homology modules*

$$H_i(I; R)$$

are Cohen–Macaulay as R/I-modules for all i.

Remark 1.17. If $I = (f_1, \dots, f_s)$, then $H_i(I; R)$ denotes the i-th homology of the Koszul complex on $f_1, \dots, f_s$. The condition that all Koszul homology modules are Cohen–Macaulay is quite restrictive and provides an effective obstruction to being licci.

1.5.2 Sufficient Conditions for Being Licci

Question 1.18. What are useful sufficient conditions for an ideal to be licci?

The notes list the following standard classes.

Theorem 1.19 (Codimension two). *Let $I \subseteq R$ have codimension 2. Then*

$$I \text{ is licci} \qquad \Longleftrightarrow \qquad R/I \text{ is Cohen–Macaulay.}$$

Remark 1.20. This is one of the fundamental starting points of linkage theory. Perfect ideals of grade two are governed by the Hilbert–Burch theorem, and repeated linkage can reduce such ideals to complete intersections.

Theorem 1.21 (Watanabe). *Let $I \subseteq R$ have codimension 3. If R/I is Gorenstein, then I is licci.*

Proposition 1.22. *Let*

$$c = \operatorname{codim} I, \qquad \delta = \mu(I) - c,$$

where $\mu(I)$ is the minimal number of generators of I. Let

$$r = \operatorname{type}(R/I),$$

the Cohen–Macaulay type, i.e. the number of minimal generators of the canonical module $\omega_{R/I}$. If

$$\delta = 1,$$

then I is licci.

Remark 1.23. The integer $\delta = \mu(I) - c$ measures how far I is from being a complete intersection. A complete intersection has $\delta = 0$. The case $\delta = 1$ is the almost complete intersection case.

Theorem 1.24 (Guerrieri–Ni–Weyman)**.** *Let $c = 3$. Under the hypotheses considered by Guerrieri–Ni–Weyman, certain small values of the type r and deviation δ force I to be licci. The lecture notes record the following numerical conditions:*

$$(r \leq 2 \text{ and } \delta \leq 4) \qquad \text{or} \qquad (r \leq 4 \text{ and } \delta \leq 4).$$

Conjecture 1.25. *Let I be Cohen–Macaulay of codimension c, type r, and deviation*

$$\delta = \mu(I) - c.$$

If

$$\frac{1}{c-1} + \frac{1}{r+1} + \frac{1}{\delta+1} > 1,$$

then I is licci.

Remark 1.26. The displayed inequality resembles the numerical inequalities that classify spherical triples. In this context it suggests that small codimension, small type, and small deviation should force the ideal into the linkage class of a complete intersection.

Conjecture 1.27 (Jelisiejew–Ramkumar–Samartano)**.** *Let I be a finite-length ideal, and suppose that*

$$[I] \in \mathrm{Hilb}_{\lambda}^{\mathrm{SM}}(\mathbb{A}^3).$$

Then $[I]$ is a smooth point of the Hilbert scheme if and only if I is licci. This is known to be true if I is a monomial ideal.

Remark 1.28. The notes emphasize that one can often extract information about whether an ideal is licci from its Betti table. However, later examples show that the Betti table alone does not completely determine licci-ness.

1.6 $\mathfrak{m}$-Primary Monomial Ideals

Let I be an $\mathfrak{m}$-primary monomial ideal in

$$R = K[x_1, \dots, x_n].$$

Then I contains pure powers of all variables. We may write

$$I = (x_1^{a_1}, \ldots, x_n^{a_n}) + I^*,$$

where I^* is the non-pure part of I, meaning the ideal generated by those minimal monomial generators involving at least two variables.

Theorem 1.29 (Huneke–Ulrich). *Let*

$$I = (x_1^{a_1}, \ldots, x_n^{a_n}) + I^*$$

be an $\mathfrak{m}$-primary monomial ideal.

(a) *If*

$$\operatorname{codim} I^* \geq 2,$$

then I is not licci.

(b) *If*

$$\operatorname{codim} I^* = 1,$$

write

$$I^* = x_1^{b_1} \cdots x_n^{b_n} K$$

for a monomial ideal K. Then I is doubly linked to

$$(x_1^{a_1-b_1}, \ldots, x_n^{a_n-b_n}) + K.$$

This process can be repeated.

Remark 1.30. Part (b) gives an algorithmic reduction procedure for certain monomial ideals. The phrase "doubly linked" means that the new ideal is obtained from I by two successive direct links. Therefore I and the reduced ideal lie in the same even liaison class.

Example 1.31. Let

$$I = (a^4, b^4, c^4, ab^2c^2, b^2c^3, a^3b^2) \subseteq K[a, b, c].$$

Factoring out common powers in the non-pure part and applying the Huneke–Ulrich reduction, the notes indicate that I is doubly linked, written

$$I \sim\sim K,$$

to the ideal

$$K = (a^3, b^2, c^3, ac^2).$$

Since K is licci, it follows that I is licci.

Remark 1.32. The conclusion follows because being licci is invariant under linkage class. If I is linked, or doubly linked, to an ideal K that is licci, then I is also licci.

Example 1.33 (Adam Boucher)**.** Let

$$J = (a^4, b^4, c^4, b^2c^3, a^2c^3, a^2b^3) \subseteq K[a, b, c].$$

This ideal is not licci.

Remark 1.34. The ideals

$$I = (a^4, b^4, c^4, ab^2c^2, b^2c^3, a^3b^2)$$

and

$$J = (a^4, b^4, c^4, b^2c^3, a^2c^3, a^2b^3)$$

have the exact same Betti table, but I is licci whereas J is not licci. Hence the Betti table alone does not determine whether an ideal is licci.

1.7 Broken Complete Intersections

Consider an ideal of the form

$$I = (x_1^{a_1}, \dots, x_n^{a_n}) + x_1K.$$

Taking the colon ideal with respect to x_1, we obtain

$$(I : x_1) = (x_1^{a_1-1}, x_2^{a_2}, \dots, x_n^{a_n}) + K.$$

Multiplication by x_1 gives the short exact sequence

$$0 \longrightarrow R/(I : x_1) \xrightarrow{\cdot x_1} R/I \longrightarrow R/(I, x_1) \longrightarrow 0.$$

The rightmost quotient is a complete intersection:

$$R/(I, x_1) \cong R/(x_1, x_2^{a_2}, \dots, x_n^{a_n}).$$

Remark 1.35. The exactness follows from the standard colon sequence. For any ideal I and element $x \in R$, there is an exact sequence

$$0 \longrightarrow R/(I:x) \xrightarrow{\cdot x} R/I \longrightarrow R/(I,x) \longrightarrow 0.$$

This sequence is a basic tool for inductive arguments on monomial ideals and for decomposing quotients into simpler pieces.

Definition 1.36 (Broken complete intersection)**.** A quotient R/I is called a *broken complete intersection* if there exists a chain of ideals

$$R \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_s = I$$

such that the successive quotients are complete intersections of codimension c. More precisely, the lecture notes indicate that the J_k are principal ideals modulo I, and that

$$J_{i-1}/J_i \cong R/(\text{regular sequence})$$

for each i.

Theorem 1.37. *For broken complete intersections, the following implications hold:*

$$\textit{broken complete intersection} \Longrightarrow \textit{licci},$$

and, for $\mathfrak{m}$*-primary monomial ideals,*

$$\mathfrak{m}\textit{-primary monomial licci} \Longrightarrow \textit{broken complete intersection.}$$

Remark 1.38. Thus, among $\mathfrak{m}$-primary monomial ideals, the broken complete intersection condition gives a structural characterization of licci behavior. The colon sequence above explains why such ideals can often be decomposed into complete intersection pieces.

1.8 Summary of Main Themes

The lecture develops three intertwined ideas:

(1) Linkage is defined by taking residual ideals with respect to complete intersections:

$$J = (\underline{\alpha}) : I, \qquad I = (\underline{\alpha}) : J.$$

(2) Licci ideals are precisely those ideals that can be linked, in finitely many steps, to a

complete intersection.

(3) For $\mathfrak{m}$-primary monomial ideals, the non-pure part I^* controls licci-ness. The Huneke–Ulrich criterion gives both non-licci obstructions and a double-link reduction procedure.

The examples of
$$I = (a^4, b^4, c^4, ab^2c^2, b^2c^3, a^3b^2)$$
and
$$J = (a^4, b^4, c^4, b^2c^3, a^2c^3, a^2b^3)$$
show that two ideals can share the same Betti table while having different linkage behavior. This illustrates that licci-ness is subtler than the numerical data encoded by the minimal free resolution.

2 Multiplicities and Reductions by Dan Katz

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. In the presence of two filtrations, one is naturally led to a bigraded Hilbert function of the form
$$H(r, s).$$
For $r, s \gg 0$, such functions often agree with a polynomial whose top-degree part can be written in the binomial-coefficient basis as
$$H(r, s) = \sum_{i+j=d} c_{ij} \binom{r}{i} \binom{s}{j} + \text{terms of total degree } < d.$$
Equivalently, in a slightly more complete polynomial form,
$$H(r, s) = \sum_{i+j \le d} c_{ij} \binom{r}{i} \binom{s}{j}.$$
The coefficients c_{ij} in total degree d are precursors to what are later called *mixed multiplicities*.

Remark 2.1. The use of binomial coefficients rather than ordinary monomials is standard in Hilbert polynomial theory. It makes the coefficients integral in many natural situations and is compatible with taking finite differences. The top-degree coefficients encode asymptotic intersection-theoretic data.

2.1 Historical Background

2.1.1 Chevalley: Early Multiplicity Formulas

Chevalley's work from the 1940s provided an early structural motivation for multiplicity. Let R be a complete local domain containing a coefficient field K, and let

$$\underline{x} = x_1, \ldots, x_d$$

be a system of parameters. In the finite-extension situation, one has a formula of the form

$$[R : K] = e(\underline{x})[R : K],$$

where $e(\underline{x})$ is the multiplicity associated to the parameter ideal $(\underline{x})$.

Remark 2.2. More precisely, one usually interprets such expressions through finite module ranks or degrees of fraction-field extensions after passing to suitable finite extensions. The displayed formula is a historical precursor to the later Hilbert–Samuel definition of multiplicity.

Chevalley also extended the philosophy to generalized local rings. In this setting, one encounters a decomposition formula of the form

$$e(\underline{x}; R) = \sum_{\mathfrak{p}} e\left(\underline{x}; \widehat{R}/\mathfrak{p}\right) \lambda_R\left(\widehat{R}_{\mathfrak{p}}\right),$$

where the sum ranges over the relevant prime ideals $\mathfrak{p}$ of $\widehat{R}$, typically the minimal primes of maximal dimension.

Remark 2.3. This formula expresses multiplicity as a sum over components of the completion. The length factor $\lambda_R(\widehat{R}_{\mathfrak{p}})$ records the scheme-theoretic thickness of the component, while $e(\underline{x}; \widehat{R}/\mathfrak{p})$ records the multiplicity along that component.

2.1.2 Samuel: Hilbert–Samuel Multiplicity

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Samuel proved that for $n \gg 0$,

$$\lambda_R(R/I^{n+1}) = P_I(n),$$

where $P_I(n)$ is a polynomial of degree d. Its leading term has the form

$$P_I(n) = \frac{e(I)}{d!} n^d + \text{lower-degree terms}.$$

The integer $e(I)$ is the *Hilbert–Samuel multiplicity* of I. If $\underline{x} = x_1, \ldots, x_d$ is a system of parameters and

$$(\underline{x}) \subseteq I,$$

then one has

$$e(\underline{x}) \geq e(I).$$

Remark 2.4. The dimension d of a Noetherian local ring $(R, \mathfrak{m})$ is also the least number of elements required to generate an $\mathfrak{m}$-primary ideal. Such a generating sequence is a system of parameters.

2.1.3 Serre: Multiplicity as an Euler Characteristic

Serre related multiplicity to homological algebra. Let

$$\underline{x} = x_1, \ldots, x_d$$

be a system of parameters. Then the multiplicity of the parameter ideal $(\underline{x})$ is the Euler characteristic of the Koszul complex $K(\underline{x}; R)$:

$$e(\underline{x}; R) = \chi(K(\underline{x}; R)) = \sum_{i=0}^{d} (-1)^i \lambda_R \left(H_i(K(\underline{x}; R)) \right).$$

Remark 2.5. When R is Cohen–Macaulay and $\underline{x}$ is a system of parameters, the Koszul homology $H_i(K(\underline{x}; R))$ vanishes for $i > 0$. Thus in that case

$$e(\underline{x}; R) = \lambda_R(R/(\underline{x})).$$

In general, the higher Koszul homology modules correct the naive length formula.

2.1.4 Northcott–Rees: Reductions and Integral Closure

Northcott and Rees introduced the notion of reductions of ideals.

Definition 2.6 (Reduction of an ideal)**.** Let $J \subseteq I$ be ideals in a Noetherian ring R. The ideal J is called a *reduction* of I if there exists an integer $n \geq 0$ such that

$$JI^n = I^{n+1}.$$

A reduction minimal with respect to inclusion is called a *minimal reduction*.

Reductions and minimal reductions exist under standard hypotheses, for example for $\mathfrak{m}$-primary ideals in a Noetherian local ring with infinite residue field. A central fact is that reductions preserve integral closure:

$$J \text{ is a reduction of } I \quad \Longrightarrow \quad \overline{J} = \overline{I}.$$

Definition 2.7 (Integral closure of an ideal)**.** Let $I \subseteq R$ be an ideal. An element $x \in R$ is integral over I if it satisfies an equation

$$x^n + i_1 x^{n-1} + \cdots + i_{n-1} x + i_n = 0, \qquad i_j \in I^j.$$

The set of all such elements is the *integral closure* of I, denoted $\overline{I}$.

2.2 The Generalization Program

The lecture traces a sequence of generalizations:

$$\begin{tikzcd}
 & \text{many ideals} \arrow[dr] & \\
\text{single ideal} \arrow[ur] \arrow[dr] & & \text{many modules} \\
 & \text{single module} \arrow[ur] &
\end{tikzcd}$$

The guiding principle is that the classical theory of Hilbert–Samuel multiplicity and reductions for one ideal should extend first to several ideals, then to a single module, and finally to several modules.

2.3 Many Ideals

2.3.1 Bigraded and Multigraded Hilbert Polynomials

Let $I_1, \ldots, I_q$ be $\mathfrak{m}$-primary ideals in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d. Risler and Teissier proved that for $n_1, \ldots, n_q \gg 0$,

$$\lambda_R\left(\frac{R}{I_1^{n_1} \cdots I_q^{n_q}}\right) = P(n_1, \ldots, n_q),$$

where P is a polynomial of total degree d. The top-degree part of P can be written as

$$\sum_{k_1 + \cdots + k_q = d} \frac{e(I_1^{[k_1]}, \ldots, I_q^{[k_q]})}{k_1! \cdots k_q!} n_1^{k_1} \cdots n_q^{k_q}.$$

The integers

$$e(I_1^{[k_1]}, \dots, I_q^{[k_q]})$$

are called the *mixed multiplicities* of $I_1, \dots, I_q$.

Remark 2.8. The notation $I_j^{[k_j]}$ means that the ideal I_j occurs k_j times in the mixed multiplicity. Thus, for example,

$$e(I^{[d]}) = e(I)$$

is the ordinary Hilbert–Samuel multiplicity.

2.3.2 The Most Important Case: (q=d)

When $q = d$, the coefficient of the squarefree monomial

$$n_1 n_2 \cdots n_d$$

is especially important. One writes

$$e(I_1, \dots, I_d)$$

for this mixed multiplicity. In other words,

$$e(I_1, \dots, I_d)$$

is the coefficient of $n_1 \cdots n_d$ in the top-degree part of the multigraded Hilbert polynomial.

Remark 2.9. The case $q = 2$ was studied earlier by Bhattacharya. The general case was developed by Risler–Teissier and later connected deeply with reductions by Rees.

2.3.3 Rees: Joint Reductions

Rees introduced the notion of joint reductions for several ideals.

Definition 2.10 (Joint reduction for ideals)**.** Let $I_1, \dots, I_q$ be ideals of R, and choose elements

$$x_i \in I_i \qquad (1 \le i \le q).$$

The sequence $x_1, \dots, x_q$ is called a *joint reduction* of $I_1, \dots, I_q$ if

$$x_1 I_2 \cdots I_q + I_1 x_2 I_3 \cdots I_q + \cdots + I_1 \cdots I_{q-1} x_q$$

is a reduction of the product

$$I_1 \cdots I_q.$$

Equivalently,

$$\sum_{i=1}^{q} x_i I_1 \cdots \widehat{I_i} \cdots I_q$$

is a reduction of $I_1 \cdots I_q$.

For $q = d$, Rees proved that if $x_1, \ldots, x_d$ is a joint reduction of $I_1, \ldots, I_d$, then

$$e(I_1, \ldots, I_d) = e(x_1, \ldots, x_d) = \chi(K(x_1, \ldots, x_d; R)).$$

Remark 2.11. This theorem is the multigraded analogue of Serre's formula for the multiplicity of a parameter ideal. The role of a single system of parameters is now played by a joint reduction with one element chosen from each ideal.

2.4 A Single Module

2.4.1 Buchsbaum–Rim Multiplicity

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let F be a free R-module of rank r. Let

$$M \subseteq F$$

be a submodule such that

$$\lambda_R(F/M) < \infty.$$

For each n, the inclusion $M \subseteq F$ induces a map on symmetric powers

$$\mathrm{Sym}_n(M) \longrightarrow \mathrm{Sym}_n(F).$$

The image of this map is denoted

$$R_n(M) \subseteq \mathrm{Sym}_n(F).$$

Equivalently, $R_n(M)$ is the degree-n part of the Rees algebra of M inside $\mathrm{Sym}(F)$.

For $n \gg 0$,

$$\lambda_R\left(\frac{\mathrm{Sym}_n(F)}{R_n(M)}\right) = P(n),$$

where $P(n)$ is a polynomial of degree

$$d + r - 1.$$

Its leading term has the form

$$\frac{\mathrm{br}(M)}{(d+r-1)!} n^{d+r-1}.$$

The integer $\operatorname{br}(M)$ is the *Buchsbaum–Rim multiplicity* of M.

Remark 2.12. When $r = 1$, a submodule $M \subseteq F \cong R$ is simply an ideal of finite colength. In that case the Buchsbaum–Rim multiplicity recovers Hilbert–Samuel multiplicity.

2.4.2 Parameter Modules and the Buchsbaum–Rim Complex

A submodule $M \subseteq F$ is called a *parameter module* when

$$\lambda_R(F/M) < \infty \qquad \text{and} \qquad \mu(M) = d + r - 1.$$

In this case, Buchsbaum and Rim showed that $\operatorname{br}(M)$ can be expressed as the Euler characteristic of the Buchsbaum–Rim complex associated to a presentation of M.

Remark 2.13 (Added Context). This is the module-theoretic analogue of Serre's formula

$$e(\underline{x}) = \chi(K(\underline{x}; R)).$$

The Koszul complex is replaced by the Buchsbaum–Rim complex.

2.4.3 Rees: Reductions of Modules

Following Rees, write

$$M^n := R_n(M)$$

for the degree-n part of the Rees algebra of M.

Definition 2.14 (Reduction of a module)**.** Let $N \subseteq M \subseteq F$ be submodules. The submodule N is called a *reduction* of M if there exists an integer $n \geq 0$ such that

$$NM^n = M^{n+1}.$$

Here the product is taken inside the Rees algebra of M in $\operatorname{Sym}(F)$.

The integral closure of a module is also defined through the Rees algebra. Let

$$R(M) \subseteq \operatorname{Sym}(F)$$

be the Rees algebra of M. Then

$$\overline{M}$$

is defined to be the degree-one component of the integral closure

$$\overline{R(M)}$$

inside $\mathrm{Sym}(F)$.

2.5 Many Modules

2.5.1 Multigraded Buchsbaum–Rim Functions

Let

$$M_i \subseteq F_i \qquad (1 \leq i \leq q)$$

be submodules of free R-modules, with

$$\lambda_R(F_i/M_i) < \infty$$

for each i.

Under suitable hypotheses, one studies multigraded length functions of the form

$$\lambda_R\left(\frac{F_1^{n_1}\cdots F_q^{n_q}}{M_1^{n_1}\cdots M_q^{n_q}}\right),$$

which agree for $n_1, \dots, n_q \gg 0$ with a polynomial. The top-degree coefficients define mixed Buchsbaum–Rim multiplicities.

Remark 2.15. The notation $F_i^{n_i}$ and $M_i^{n_i}$ is interpreted inside the corresponding symmetric or Rees algebras. In the case where all F_i are the same free module F, these products live naturally inside $\mathrm{Sym}(F)$.

2.5.2 Callejas-Bedregal–Perez: Several Modules in One Free Module

Let

$$M_1, \dots, M_q \subseteq F$$

be submodules of a fixed free module F of rank r, and suppose

$$\lambda_R(F/M_i) < \infty$$

for all i. The associated length function is

$$\lambda_R\left(\frac{F^{n_1+\cdots+n_q}}{M_1^{n_1}\cdots M_q^{n_q}}\right) = P(n_1, \dots, n_q), \qquad n_i \gg 0.$$

The degree of this polynomial is

$$d + r - 1.$$

If $x_i \in M_i$, then the sequence $x_1, \ldots, x_q$ is a joint reduction if

$$\sum_{i=1}^{q} x_i M_1 \cdots \widehat{M_i} \cdots M_q$$

is a reduction of

$$M_1 \cdots M_q.$$

2.5.3 Several Modules in Several Free Modules

More generally, let

$$M_1 \subseteq F_1, \ldots, M_q \subseteq F_q$$

with

$$\lambda_R(F_i/M_i) < \infty$$

for every i. Write

$$r_i = \operatorname{rank}(F_i).$$

One studies the corresponding multigraded Hilbert polynomial

$$\lambda_R\left(\frac{F_1^{n_1} \cdots F_q^{n_q}}{M_1^{n_1} \cdots M_q^{n_q}}\right) = P(n_1, \ldots, n_q), \qquad n_i \gg 0.$$

The total degree is

$$d + r_1 + \cdots + r_q - q.$$

Remark 2.16. This degree formula simultaneously generalizes the degree d of the Hilbert–Samuel polynomial for ideals and the degree $d+r-1$ of the Buchsbaum–Rim polynomial for one module in a free module of rank r.

2.6 Joint Reductions of Modules

Definition 2.17 (Joint reduction for modules)**.** Let

$$M_i \subseteq F_i \qquad (1 \le i \le q)$$

be submodules of free modules. Let

$$B_i \subseteq M_i$$

be submodules. The family

$$B_1, \ldots, B_q$$

is called a *joint reduction* of

$$M_1, \dots, M_q$$

if

$$\sum_{i=1}^{q} B_i M_1 \cdots \widehat{M_i} \cdots M_q$$

is a reduction of

$$M_1 \cdots M_q.$$

Theorem 2.18 (DKV: Existence and valuation criteria for joint reductions)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let*

$$M_i \subseteq F_i \qquad (1 \le i \le q)$$

be submodules of finite colength in free modules. Then the following statements hold.

(1) Joint reductions exist if $\operatorname{depth} R > 0$.

(2) The submodules B_i may be chosen to be free.

(3) In the case $d = q$, the number of generators of the chosen reduction data is governed by the ranks of the ambient modules.

(4) The following are equivalent:

(i) $B_1, \dots, B_q$ is a joint reduction of $M_1, \dots, M_q$.

(ii) For all minimal primes P of R and all discrete valuation rings

$$R/P \subseteq V \subseteq \operatorname{Qf}(R/P),$$

the extensions of the corresponding modules to V agree:

$$B_i V = M_i V \qquad \textit{for the relevant indices } i.$$

(5) The determinant ideals

$$\det(B_1), \dots, \det(B_q)$$

form a joint reduction of the determinant ideals

$$I_1, \dots, I_q, \qquad I_j = I(M_j).$$

Remark 2.19. The valuation criterion is the module-theoretic analogue of the valuative criterion for integral closure of ideals. Passing to determinant ideals translates module-theoretic reduction questions into ideal-theoretic reduction questions.

2.7 The General Mixed Buchsbaum–Rim Theory

The general length function has the form

$$\lambda_R\left(\frac{F_1^{n_1}\cdots F_q^{n_q}}{M_1^{n_1}\cdots M_q^{n_q}}\right) = P(n_1,\ldots,n_q), \qquad n_i \gg 0.$$

Its total degree is

$$d + r_1 + \cdots + r_q - q.$$

If

$$q = d,$$

then

$$\deg P = r_1 + \cdots + r_d.$$

In this case, assuming

$$B_i \subseteq M_i,$$

one defines the mixed Buchsbaum–Rim multiplicity by extracting the coefficient of

$$n_1^{r_1}\cdots n_d^{r_d}.$$

Definition 2.20 (Mixed Buchsbaum–Rim multiplicity)**.** When $q = d$, define

$$\mathrm{br}(M_1,\ldots,M_d) = r_1!\cdots r_d!\cdot\left(\text{coefficient of } n_1^{r_1}\cdots n_d^{r_d} \text{ in } P(n_1,\ldots,n_d)\right).$$

2.7.1 Complexes Attached to the Reduction Data

Let

$$\phi_i : B_i \longrightarrow F_i$$

denote the map induced by the inclusion $B_i \subseteq M_i \subseteq F_i$. The notation

$$K(\phi_i)$$

denotes the appropriate Buchsbaum–Rim-type complex attached to ϕ_i.

One then forms the tensor product complex

$$K(\phi) = \bigotimes_i K(\phi_i).$$

The source notes contain the schematic exact sequence

$$0 \longrightarrow F_j \longrightarrow F_i \longrightarrow 0 \qquad K(\phi_i).$$

Theorem 2.21 (DKV: Euler characteristic formula)**.** *With notation as above,*

$$\mathrm{br}(M_1, \dots, M_q) = \chi(K(\phi)) = e(I_1, \dots, I_d),$$

where

$$K(\phi) = \bigotimes_i K(\phi_i)$$

and

$$I_i = I(M_i)$$

are the corresponding determinant ideals.

Remark 2.22. This formula simultaneously generalizes three classical identities:

$$e(\underline{x}) = \chi(K(\underline{x}; R)),$$

the Buchsbaum–Rim Euler characteristic formula for parameter modules, and Rees's formula for mixed multiplicities of ideals via joint reductions.

2.8 Case Study: Complete Modules over a Two-Dimensional Regular Local Ring

Let R be a two-dimensional regular local ring. Let

$$M_1 \subseteq F_1, \qquad M_2 \subseteq F_2$$

be complete modules, and let

$$B_i \subseteq M_i \qquad (i = 1, 2)$$

form a joint reduction. Then one has the identity

$$M_1 M_2 = M_1 B_2 + B_1 M_2.$$

This gives an explicit description of the length function

$$\lambda_R\left(\frac{F_1^{n_1} F_2^{n_2}}{M_1^{n_1} M_2^{n_2}}\right).$$

Remark 2.23. In dimension two, complete ideals and complete modules often admit especially concrete product formulas. The displayed equality says that the product M_1M_2 is controlled by the joint reduction data. Consequently, higher products can often be expanded recursively in terms of B_1 and B_2, leading to explicit formulas for the corresponding Hilbert function.

Summary Diagram

The lecture may be summarized by the following conceptual diagram:

$$\begin{tikzcd}
\text{Hilbert–Samuel multiplicity} \arrow[r, "\text{many ideals}"] \arrow[d, "\text{single module}"'] & \text{mixed multiplicities} \arrow[d, "\text{joint reductions}"] \\
\text{Buchsbaum–Rim multiplicity} \arrow[r, "\text{many modules}"'] & \text{mixed Buchsbaum–Rim multiplicities}
\end{tikzcd}$$