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1. Lecture 1 — Linkage and Licci Ideals By Craig Huneke

Notation and Setup

Let $(R, \mathfrak{m})$ be a regular local ring, or let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$.

Let $I \leq R$ be an ideal. In the homogeneous case, consider the minimal free resolution of R/I:

$$\cdots \longrightarrow \bigoplus_j R(-d_{cj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(-d_{1j}) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where c is the codimension of I. We denote the maximum and minimum degree shifts in the i-th step of the resolution as:

$$D_i = \max_j \{d_{ij}\} \quad \text{and} \quad d_i = \min_j \{d_{ij}\}.$$

Let I be a proper ideal. Consider a regular sequence $\alpha_1, \alpha_2, \dots, \alpha_c$ inside I. We denote the ideal generated by this sequence as $\underline{\alpha} = (\alpha_1, \dots, \alpha_c)$.

Definition 1.1 (Linkage)**.** Let $I, J \subsetneq R$ be proper ideals of the same codimension c. We say that I and J are linked by the regular sequence $\underline{\alpha} = \alpha_1, \dots, \alpha_c \subseteq I \cap J$ if

$$J = (\underline{\alpha}) : I \quad \text{and} \quad I = (\underline{\alpha}) : J.$$

In this case we write $I \sim_{\underline{\alpha}} J$.

Definition 1.2 (Peskine-Szpiro)**.** We say two proper ideals I and J are linked via a regular sequence $\underline{\alpha}$ if:

$$\underline{\alpha} : I = J \quad \text{and} \quad \underline{\alpha} : J = I.$$

Definition 1.3. Two ideals I_0 and I_l are in the same **linkage class** if there exists a sequence of linkages:

$$I_0 \sim_{\alpha_0} I_1 \sim_{\alpha_1} \cdots \sim_{\alpha_{l-1}} I_l.$$

Linkage via resolutions: If I is linked to J via a complete intersection $(\underline{\alpha}) = (\alpha_1, \dots, \alpha_c) \subseteq I$, then $J = (\underline{\alpha}) : I$. A comparison map from a free resolution of R/I to the Koszul complex on $\underline{\alpha}$ yields, via a mapping cone, a free resolution of R/J.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_c & \longrightarrow \cdots \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow \cdots \longrightarrow & R^c & \longrightarrow & R/\underline{\alpha} & \longrightarrow & 0 \end{array}$$

$$J = \underline{\alpha} : I = I_1(a) + (\alpha_1, \dots, \alpha_c)$$

Alternatively, $J = I^\vee$ (Alexander Dual).

Remark 1.4. In the special setting of squarefree monomial ideals, Alexander duality is a different construction that is often useful for studying resolutions and combinatorial properties. It should not be identified with linkage in general.

Example 1.5. Consider the ideal $(x_1^2, \dots, x_n^2, J)$ where J is a square-free monomial ideal. Taking Frobenius powers $x^{[q]}$ relates ideals of this form.

Remark 1.6 (Hartshorne–Rao)**.** For a locally Cohen–Macaulay curve $C \subseteq \mathbb{P}^3$, the Hartshorne–Rao module

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t))$$

is, up to shift, an invariant of the even liaison class of C.

Licci Ideals

Definition 1.7 (Licci)**.** An ideal I is **licci** if it is in the linkage class of a complete intersection (i.e., a regular sequence).

Several necessary conditions must hold for an ideal to be licci:

Theorem 1.8 (Buchweitz)**.** *If an ideal I is licci, then I is strongly non-obstructed, i.e., $J/I^2 \otimes \omega_{R/I}$ is a Cohen-Macaulay module over R/I.*

Theorem 1.9 (Huneke)**.** *If an ideal I is licci, then I is strongly Cohen-Macaulay. In particular, the Koszul homology groups $H_i(I; R)$ are Cohen-Macaulay over R/I.*

Sufficient Conditions for Licci: What about sufficient conditions for an ideal to be licci?

1) **Codimension 2:** If $\operatorname{codim} I = 2$, then I is licci $\iff$ R/I is Cohen-Macaulay.
2) **Codimension 3:** If $\operatorname{codim} I = 3$ and R/I is Gorenstein, then I is licci (J. Watanabe).

   Alternatively, let $c = \operatorname{codim} I$, and define $\delta = \mu(I) - c$, where $\mu(I)$ is the minimal number of generators. Let $r = \operatorname{type}(R/I)$ (the number of generators in the canonical module). If $\delta = 1$, then I is licci.
3) **Guerrieri-Ni-Weyman:** If $c = 3$ and $Q \subseteq R$, then if ($r \leq 2$ and $\delta \leq 4$) or ($r \leq 4$ and $\delta \leq 4$), then I is licci.

Conjecture 1.10. If I is Cohen-Macaulay, then the following inequality implies I is licci:

$$\frac{1}{c-1} + \frac{1}{r+1} + \frac{1}{\delta+1} > 1 \implies I \text{ is licci.}$$

Conjecture 1.11 (Jelesiejew-Ramkumar-Samartano)**.** If I is of finite length, and $[I] \in \operatorname{Hilb}_\lambda^{SM}(\mathbb{A}^3)$, then $[I]$ is a smooth point if and only if I is licci. This is known to be true if I is a monomial ideal. (We can get information about an ideal being licci from its Betti table).

$\mathfrak{m}$-primary Monomial Ideals

Let I be an $\mathfrak{m}$-primary monomial ideal. We can write I in the form:

$$I = (x_1^{a_1}, \ldots, x_n^{a_n}) + I^*$$

where I^* is the non-pure part.

Theorem 1.12 (Huneke-Ulrich). *(a) If* $\operatorname{codim} I^* \geq 2$*, then* I *is not licci.*

(b) If $\operatorname{codim} I^* = 1$*, write* $I^* = x_1^{b_1} \cdots x_n^{b_n} K$*. Then* I *is doubly linked to the ideal* $(x_1^{a_1-b_1}, \ldots, x_n^{a_n-b_n}) + K$*. This process can be repeated.*

Example 1.13. Let $I = (a^4, b^4, c^4, ab^2c^2, b^2c^3, a^3b^2)$. We factor out common powers and find that I is doubly linked ($\sim\sim$) to:

$$K = (a^3, b^2, c^3, ac^2).$$

Since K is licci, it follows that I is licci.

Example 1.14 (Adam Boucher). Let $J = (a^4, b^4, c^4, b^2c^3, a^2c^3, a^2b^3)$. This ideal is *not* licci.

Remark 1.15. The ideals I and J in the two examples above have the exact same Betti table, showing that the Betti table alone does not completely determine if an ideal is licci.

Broken Complete Intersections

Consider an ideal of the form $I = (x_1^{a_1}, \ldots, x_n^{a_n}) + x_1 K$. Taking the colon ideal with respect to x_1, we get:

$$(I : x_1) = (x_1^{a_1-1}, x_2^{a_2}, \ldots, x_n^{a_n}) + K$$

This yields a short exact sequence:

$$0 \longrightarrow R/(I : x_1) \longrightarrow R/I \longrightarrow R/(I, x_1) \longrightarrow 0$$

Notice that the rightmost term is a complete intersection (C.I.):

$$R/(I, x_1) \cong R/(x_1, x_2^{a_2}, \ldots, x_n^{a_n}).$$

Definition 1.16. A quotient R/I is a **broken complete intersection** if there exists a chain of ideals:

$$R \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_s = I$$

where the J_k are principal ideals modulo I, such that successive quotients J_{i-1}/J_i are complete intersections of codimension c. That is:

$$J_{i-1}/J_i \cong R/(\text{regular sequence}).$$

Theorem 1.17. *The following implications hold for broken complete intersections:*

- *Broken C.I.* $\Longrightarrow$ *licci.*
- *Broken C.I.* $\Longleftarrow$ $\mathfrak{m}$*-monomial licci.*

2. Lecture 2 — Multiplicities & Reductions by Dan Katz

Bigraded Hilbert Function

The bigraded Hilbert function is given by:

$$H(r, s) = \sum_{i+j=d} c_{ij} \binom{r}{i} \binom{s}{j}$$

The coefficients c_{ij} are precursors to "mixed multiplicities".

Historical Context

(1943, 1945) Chevalley

1. Let R be a complete local domain, $K \subseteq R$, with $[R : K] < \infty$, and let $\underline{x}$ be a system of parameters (s.o.p.). The degree of the field extension provides a foundation:

   $$[R : K[\underline{x}]] = e(\underline{x})[R : K]$$

   This formulation naturally leads to the definition of multiplicities.

2. Extending this to Generalized Local Rings (GLR), the multiplicity is calculated as:

   $$e(\underline{x}) = \sum e\left(\underline{x}\frac{\hat{R}}{\mathfrak{p}}\right) \lambda(\hat{R}_{\mathfrak{p}})$$

Chevalley (historical precursor):
Early work related multiplicity to finite extensions determined by systems of parameters, helping motivate the later formal definition of multiplicity.
Samuel (1947, 1949, 1951):
Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then for $n \gg 0$,

$$\lambda(R/I^{n+1}) = P_I(n),$$

where $P_I(n)$ is a polynomial of degree d. Its leading term is

$$P_I(n) = \frac{e(I)}{d!} n^d + \text{lower degree terms},$$

where $e(I)$ is the Hilbert–Samuel multiplicity of I. If $\underline{x}$ is a system of parameters with $(\underline{x}) \subseteq I$, then one has

$$e(\underline{x}) \geq e(I).$$

Here, d is defined as the least number of elements generating an $\mathfrak{m}$-primary ideal.

Serre (early 50's):
For a system of parameters $\underline{x}$, the multiplicity can be expressed as the Euler characteristic of the corresponding Koszul complex $K(\underline{x})$:

$$\underline{x} = \text{s.o.p.}, \quad e(\underline{x}) = \chi(K(\underline{x})) = \sum_{i=0}^{d} (-1)^i \lambda(H_i(K(\underline{x})))$$

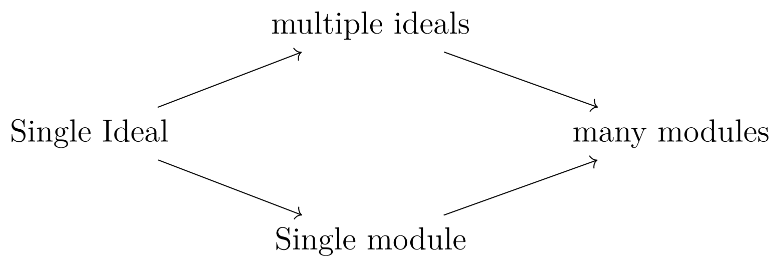
Northcott-Rees (1954):

- An ideal $J \subseteq R$ is defined as a *reduction* of I if $JI^n = I^{n+1}$ for some n.
- Reductions and minimal reductions are guaranteed to exist.
- The integral closures of the ideals match: $\overline{J} = \overline{I}$.

$$\left[x \in \overline{I} \iff x^n + i_1 x^{n-1} + \cdots + i_n = 0, \quad i_j \in I^j\right]$$

Generalizations

The goal is to generalize these concepts from single ideals to more complex structures:



1. Many Ideals

Bigraded Hilbert Polynomial

For $r, s \gg 0$, the Hilbert function agrees with a polynomial

$$H(r,s) = \sum_{i+j \leq d} c_{ij} \binom{r}{i} \binom{s}{j}.$$

Its top-degree part is

$$\sum_{i+j=d} c_{ij} \binom{r}{i} \binom{s}{j},$$

and the coefficients in top degree lead to the mixed multiplicities.
Risler–Teissier (1973):
Let $I_1, \ldots, I_q$ be $\mathfrak{m}$-primary ideals in a local ring of dimension d. Then for $n_1, \ldots, n_q \gg 0$,

$$\lambda\left(\frac{R}{I_1^{n_1} \cdots I_q^{n_q}}\right) = P(n_1, \ldots, n_q),$$

where P is a polynomial of total degree d. Its top-degree part can be written as

$$\sum_{k_1+\cdots+k_q=d} \frac{e(I_1^{[k_1]},\ldots,I_q^{[k_q]})}{k_1!\cdots k_q!}\, n_1^{k_1}\cdots n_q^{k_q}.$$

The coefficients $e(I_1^{[k_1]},\ldots,I_q^{[k_q]})$ are the mixed multiplicities.
Most Important Case ($q = d$):
Set $e(I_1 \ldots I_d) =$ coeff of $n_1 \ldots n_d$. This is "the" mixed multiplicity of $I_1,\ldots,I_d$ (the case for $n = 2$ was studied by Bhattacharya).

Rees (1984): "Joint Reductions"
Let $I_1,\ldots,I_q$ be $\mathfrak{m}$-primary ideals, and pick elements $x_i \in I_i$. The sequence $x_1 \ldots x_q$ is a *joint reduction* if:

$$x_1 I_2 \ldots I_q + I_1 \ldots I_{q-1} x_q \quad \text{is a reduction of } I_1 \ldots I_q.$$

If $\underline{q = d}$, Rees proved:

$$e(I_1 \ldots I_d) = e(\underline{x}) = \chi(K(\underline{x}))$$

2. Single Module

Buchsbaum–Rim (1963):
Let $M \subseteq F$ be a submodule of a free R-module F of rank r, and assume $\lambda(F/M) < \infty$. For each n, the natural map

$$S_n(M) \longrightarrow S_n(F)$$

has image denoted $R_n(M)$. Then for $n \gg 0$,

$$\lambda\left(\frac{S_n(F)}{R_n(M)}\right) = P(n),$$

where $P(n)$ is a polynomial of degree $d + r - 1$. Its leading term is

$$\frac{br(M)}{(d+r-1)!} n^{d+r-1},$$

where $br(M)$ is the Buchsbaum–Rim multiplicity of M.
Parameter module case:
If $\mu(M) = d + r - 1$, then the Buchsbaum–Rim multiplicity can be expressed as the Euler characteristic of the Buchsbaum–Rim complex associated to a presentation of M.

Rees (1987): Reduction of module.
Write M^n for $R_n(M)$. A submodule $N \subseteq M$ is a reduction if:

$$NM^n = M^{n+1}$$

Let $\overline{M}$ be defined as the degree 1 component of $\overline{R(M)}$ in $S(F)$.

3. Many Modules

Let $M_i \subseteq F_i$ be submodules of free modules, with $\lambda(F_i/M_i) < \infty$ for each i. Under suitable hypotheses, one studies multigraded length functions of the form

$$\lambda\left(\frac{F_1^{n_1} \cdots F_q^{n_q}}{M_1^{n_1} \cdots M_q^{n_q}}\right),$$

which agree for large $(n_1, \dots, n_q)$ with a polynomial whose top-degree coefficients define mixed Buchsbaum–Rim multiplicities.
(2015) Callejas-Bedregal + Perez:

(i) Let $M_1, \dots, M_q \subseteq F$ with $\lambda(F/M_i) < \infty$. We study the length function:

$$\lambda\left(\frac{F^{n_1+\cdots+n_q}}{M_1^{n_1} \dots M_q^{n_q}}\right) = P(n_1, \dots, n_q), \quad n_i \gg 0$$

The degree is $d + r - 1$.
Let $x_i \in M_i$. The sequence $x_1, \dots, x_q$ is a Joint Reduction (JR) if $\sum x_i M_1 \dots \widehat{M_i} \dots M_q$ is a reduction of $M_1 \dots M_q$.

(ii) Let $M_1 \subseteq F_1, \dots, M_q \subseteq F_q$ with $\lambda(F_i/M_i) < \infty$ and $r_k = \mathrm{rk}(M_k)$.

Definition 2.1 (Joint reduction)**.** Let $B_i \subseteq M_i$ for $i = 1, \dots, q$. One says that $B_1, \dots, B_q$ form a joint reduction of $M_1, \dots, M_q$ if

$$\sum_{i=1}^{q} B_i \, M_1 \cdots \widehat{M_i} \cdots M_q$$

is a reduction of $M_1 \cdots M_q$.

Theorem 2.2 (DKV)**.** *1. Joint reductions exist if* $\operatorname{depth} R > 0$. *$B_i$ can be taken to be free. If $d = q$, then $\mu(B) \sim \mu(M_i)$.*

2. The following are equivalent (TFAE):

(i) $B_1, \dots, B_q$ is a JR.

(ii) For all minimal primes P and discrete valuation rings (DVRs) $R/P \subseteq V \subseteq$ $\mathrm{QF}(R/P)$, we have $B_x V = M_x V$.

(iii) $\det B_1, \dots, \det B_q$ is a JR, comparing to $I_1, \dots, I_q$ where $I_j = I(M_j)$.

General Theory

$$\lambda\left(\frac{F_1^{n_1} \dots F_q^{n_q}}{M_1^{n_1} \dots M_q^{n_q}}\right) = P(n_1, \dots, n_q), \quad n_i \gg 0.$$

The total degree is given by $d + r_1 + \cdots + r_q - q$.

If $q = d$, then $\deg = r_1 + r_2 + \cdots + r_d$. (Assuming $B_i \subseteq M_i$).

Definition 2.3. $br(M) = r_1! \dots r_d! \cdot (\text{coefficient of } n_1^{r_1} \dots n_d^{r_d})$.

Let the following exact sequence define $K(\phi_i)$:

$$0 \to F_j \to F_i \to 0 \quad K(\phi_i)$$

Theorem 2.4 (DKV).

$$br(M_1 \dots M_q) = \chi(K(\phi)) = e(I_1 \dots I_d)$$

where $K(\phi) = \bigotimes_i K(\phi_i)$.

Case Study:
Let R be a 2-dimensional Regular Local Ring (RLR).
Let M_1, M_2 be complete modules, and $B_i \subseteq M_i$ be a Joint Reduction (JR).
Then we have the identity:

$$M_1 M_2 = M_1 B_2 + B_1 M_2$$

This leads to an **Explicit Description!** for the length function:

$$\lambda\left(\frac{F_1^{n_1} F_2^{n_2}}{M_1^{n_1} M_2^{n_2}}\right).$$