CA+ Conference

Commutative Algebra

Iowa State University

Ames, Iowa

Conference Notes

A Two-Day Conference on Commutative Algebra and Related Fields like Algebraic Geometry, Number Theory, Combinatorics, and More

Notes from the **CA+ Conference**

Contents

1 The Briançon–Skoda Theorem by Linquan Ma

Preliminaries and Background

Definition 1.1 (Integral closure of an ideal)**.** Let R be a commutative ring and let $I \subseteq R$ be an ideal. The **integral closure** of I, denoted $\overline{I}$, consists of all elements $x \in R$ for which there exists an integer $n \geq 1$ and elements $a_i \in I^i$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

Equivalently, x is integral over the Rees algebra filtration $\{I^i\}_{i \geq 0}$.

Remark 1.2. The condition $a_i \in I^i$ is the ideal-theoretic analogue of integral dependence over a subring. When $I = (f_1, \ldots, f_r)$, the statement $x \in \overline{I}$ may also be tested valuatively in many standard settings: $x \in \overline{I}$ if and only if $v(x) \geq v(I)$ for every relevant valuation v. This valuative perspective is often the quickest way to understand why integral closure appears naturally in birational geometry.

Remark 1.3 (Geometric interpretation). Under mild hypotheses, let

$$Y = \widetilde{\mathrm{Bl}_I R}$$

be the normalization of the blowup of $\mathrm{Spec}(R)$ along I, and let

$$\pi \colon Y = \widetilde{\mathrm{Bl}_I R} \longrightarrow \mathrm{Spec}(R)$$

be the resulting proper birational morphism. If E denotes the exceptional Cartier divisor determined by the invertible ideal sheaf

$$\mathcal{O}_Y(-E) = I\mathcal{O}_Y,$$

then

$$\overline{I^n} = \pi_* \mathcal{O}_Y(-nE).$$

Thus, integral closures of powers of I are recovered as pushforwards of natural line bundles on the normalized blowup.

Definition 1.4 (Pseudo-rational local ring)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. The ring R is called **pseudo-rational** if:

(i) R is Cohen–Macaulay;

(ii) R is normal;

(iii) the completion $\widehat{R}$ is reduced;

(iv) for every proper birational map

$$\pi\colon Y \longrightarrow \operatorname{Spec}(R),$$

the natural map on top local cohomology

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}\big(\mathbf{R}\pi_*\mathcal{O}_Y\big)$$

is injective.

Equivalently, in the presence of a dualizing complex and the usual hypotheses under which Grothendieck duality applies, one has

$$\pi_*\omega_Y \cong \omega_R.$$

Remark 1.5. The following standard implications and comparisons were emphasized:

- If R is regular, then R is pseudo-rational.
- If R is essentially of finite type over a field of characteristic 0, then

$$R \text{ is pseudo-rational} \quad \Longleftrightarrow \quad R \text{ has rational singularities.}$$

- If R is excellent and $\operatorname{char}(R) = p > 0$, then

$$F\text{-rational} \quad \Longrightarrow \quad \text{pseudo-rational.}$$

  The converse is not true in general.

Example 1.6 (Brieskorn hypersurface singularities)**.** Let

$$R = \frac{k[[x,y,z]]}{(x^a + y^b + z^c)}.$$

Then R is pseudo-rational if and only if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

Remark 1.7. The inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$$

is the familiar numerical condition appearing in the classification of two-dimensional Brieskorn singularities with rational behavior. The lecture used this as a concrete family illustrating that pseudo-rationality is a singularity condition weaker than regularity but still strong enough to imply Briançon–Skoda type containment theorems.

Classical Briançon–Skoda Type Results

Theorem 1.8 (Briançon–Skoda; Lipman–Sathaye)**.** *Let R be a regular ring, and let $I \subseteq R$ be an ideal generated by n elements. Then*

$$\overline{I^{n+k-1}} \subseteq I^k \qquad \textit{for all } k \geq 1.$$

In particular, taking $k = 1$ gives

$$\overline{I^n} \subseteq I.$$

Remark 1.9 (Reductions). Suppose $J \subseteq I$ is a reduction of I, meaning that

$$\overline{J} = \overline{I}.$$

If $\mu(J) = \ell$, then the Briançon–Skoda theorem applied to J gives

$$\overline{I^{\ell+k-1}} = \overline{J^{\ell+k-1}} \subseteq J^k \subseteq I^k.$$

In particular, using a minimal reduction, one obtains the dimension bound

$$\overline{I^{\dim(R)+k-1}} \subseteq I^k \qquad \text{for all ideals } I \subseteq R \text{ and all } k \geq 1,$$

provided the usual hypotheses ensuring existence of minimal reductions are in force.

Remark 1.10. Historically:

(i) The original Briançon–Skoda theorem was proved for

$$R = \mathbb{C}\{x_1, \ldots, x_d\},$$

the ring of convergent power series.

(ii) Lipman and Sathaye extended the theorem to the general regular case.

(iii) There are now proofs using multiplier ideals, test ideals, tight closure, and other methods.

Theorem 1.11 (Huneke, 1990). *Suppose R is reduced and essentially of finite type over either a field or a mixed characteristic discrete valuation ring. Then there exists an integer N, depending only on R, such that*

$$\overline{I^{N+k-1}} \subseteq I^k \qquad \text{for all ideals } I \subseteq R \text{ and all } k \geq 1.$$

For comparison, the Briançon–Skoda theorem says that one may take $N = \dim R$ when R is regular.

Theorem 1.12 (Lipman–Teissier). *Let $(R, \mathfrak{m})$ be pseudo-rational. Then, for every ideal $I \subseteq R$,*

$$\overline{I^{\dim(R)+k-1}} \subseteq I^k \qquad \text{for all } k \geq 1.$$

Warmup Examples

Example 1.13 (Warmup). Let $f, g, h \in \mathbb{C}[x, y]$. Show that

$$f^2g^2h^2 \in (f^3, g^3, h^3).$$

Proof. Set

$$J = (f^3, g^3, h^3) \subseteq \mathbb{C}[x, y].$$

We first show that $f^2g^2h^2$ is integral over J^2. Indeed,

$$(f^2g^2h^2)^3 = f^6g^6h^6 = (f^3)^2(g^3)^2(h^3)^2.$$

Since

$$(f^3)^2(g^3)^2(h^3)^2 \in J^6 = (J^2)^3,$$

the element $f^2g^2h^2$ satisfies the monic equation

$$T^3 - (f^3)^2(g^3)^2(h^3)^2 = 0$$

with constant term in $(J^2)^3$. Hence

$$f^2g^2h^2 \in \overline{J^2}.$$

Because $\mathbb{C}[x, y]$ is regular of dimension 2, the dimension form of the Briançon–Skoda theorem gives

$$\overline{J^2} \subseteq J.$$

Therefore

$$f^2g^2h^2 \in (f^3, g^3, h^3).$$

$\square$

Example 1.14. Let

$$R = \frac{\mathbb{C}[x,y,z]}{(x^2+y^4+z^4)}.$$

Then $\dim(R) = 2$. In R one has

$$x^2 = -(y^4+z^4) \in (y,z)^4.$$

Therefore

$$x \in \overline{(y,z)^2}.$$

However,

$$x \notin (y,z).$$

Thus, on this surface singularity, one can have

$$x \in \overline{(y,z)^2} \qquad \text{but} \qquad x \notin (y,z).$$

Remark 1.15. The implication

$$x^2 \in (y,z)^4 \quad \Longrightarrow \quad x \in \overline{(y,z)^2}$$

comes from the defining equation for integral dependence. Namely, x satisfies

$$T^2 - x^2 = 0,$$

and the constant term $-x^2$ lies in

$$\left((y,z)^2\right)^2 = (y,z)^4.$$

Hence x is integral over $(y,z)^2$.

Main Results

Theorem 1.16 (MMRS 2025, Theorem A)**.** *Let $I \subseteq R$ be an ideal with $\mu(I) = n$.*

(i) *If R is pseudo-rational, then*

$$\overline{I^{n+k-1}} \subseteq I^k \qquad \text{for all } k \geq 1.$$

(ii) *If R is F-pure or Du Bois, then*

$$\overline{I^{n+k}} \subseteq I^k \qquad \text{for all } k \geq 1.$$

In particular,

$$\overline{I^{\dim(R)+1}} \subseteq I$$

in the corresponding dimension-bounded form.

Theorem 1.17 (MMRS 2025, Theorem B)**.** *Let R be excellent, reduced, and of finite Krull dimension. Then there exists an integer N, depending only on R, such that*

$$\overline{I^{N+k-1}} \subseteq I^k \qquad \text{for all ideals } I \subseteq R \text{ and all } k \geq 1.$$

Proofs and Derived Splittings

Lemma 1.18 (Kovács)**.** *If R is pseudo-rational and*

$$\pi\colon Y \longrightarrow \operatorname{Spec}(R)$$

is a proper birational morphism, then the natural map

$$R \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y$$

splits in the derived category $D(R)$.

Proof sketch. The pseudo-rationality hypothesis gives the canonical identification

$$\omega_R \cong \pi_*\omega_Y$$

and, more precisely, a trace-type composition in the derived category

$$\omega_R^\bullet = \omega_R \longrightarrow \pi_*\omega_Y \longrightarrow \mathbf{R}\pi_*\omega_Y \longrightarrow \mathbf{R}\pi_*\omega_Y^\bullet \xrightarrow{\mathrm{Tr}} \omega_R^\bullet.$$

This composition is the identity on the dualizing complex $\omega_R^\bullet$. Applying the duality functor

$$\mathbf{R}\mathrm{Hom}_R(-, \omega_R^\bullet)$$

to this identity yields a splitting

$$R \longleftarrow \mathbf{R}\pi_*\mathcal{O}_Y \longleftarrow R$$

in $D(R)$. Equivalently, the composition

$$R \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y \longrightarrow R$$

is the identity. □

Remark 1.19. The key point is that pseudo-rationality is not used merely as a vanishing condition. It gives a strong trace compatibility statement for the dualizing complex. After applying Grothendieck duality, this trace compatibility becomes a derived splitting of the structure sheaf map.

Proof of the Pseudo-Rational Briançon–Skoda Containment

Proof of Theorem A in the pseudo-rational case, for $k = 1$. We prove the containment

$$\overline{I^n} \subseteq I$$

under the assumption that R is pseudo-rational and that

$$I = (f_1, \dots, f_n).$$

Let

$$\pi\colon Y = \widetilde{\mathrm{Bl}_I R} \longrightarrow \mathrm{Spec}(R)$$

be the normalized blowup of I. Let E be the exceptional divisor on Y, so that

$$\mathcal{O}_Y(-E) = I\mathcal{O}_Y.$$

Then

$$\overline{I^n} = \pi_*\mathcal{O}_Y(-nE).$$

Consider the Koszul complex on the sequence

$$\underline{f} = (f_1, \dots, f_n)$$

over R:

$$\operatorname{Kos}(\underline{f};R) = \left[0 \to R \to R^{\oplus n} \to \cdots \to R^{\oplus\binom{n}{2}} \to R^{\oplus n} \xrightarrow{(f_1,\dots,f_n)} R \to 0\right].$$

Pulling this complex back to Y gives

$$\pi^* \operatorname{Kos}(\underline{f};R) = \left[0 \to \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus n} \to \cdots \to \mathcal{O}_Y^{\oplus\binom{n}{2}} \to \mathcal{O}_Y^{\oplus n} \xrightarrow{(f_1,\dots,f_n)} \mathcal{O}_Y \to 0\right].$$

Since each f_i belongs to $I\mathcal{O}_Y = \mathcal{O}_Y(-E)$, each f_i may be viewed as a section of $\mathcal{O}_Y(-E)$. Equivalently, each f_i defines a map

$$\mathcal{O}_Y(E) \longrightarrow \mathcal{O}_Y.$$

Twisting the Koszul complex by the powers of $\mathcal{O}_Y(-E)$ gives the exact Koszul complex associated to the globally generated invertible ideal $I\mathcal{O}_Y$:

$$0 \to \mathcal{O}_Y(-nE) \to \mathcal{O}_Y(-(n-1)E)^{\oplus n} \to \cdots \to \mathcal{O}_Y(-E)^{\oplus n} \xrightarrow{(f_1,\dots,f_n)} \mathcal{O}_Y \to 0.$$

There is a natural map of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y(-nE) \arrow[r] \arrow[d] & \mathcal{O}_Y(-(n-1)E)^{\oplus n} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{O}_Y(-E)^{\oplus n} \arrow[r, "(f_1,\dots,f_n)"] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & {} \\
0 \arrow[r] & \mathcal{O}_Y \arrow[r] & \mathcal{O}_Y^{\oplus n} \arrow[r] & \cdots \arrow[r] & \mathcal{O}_Y^{\oplus n} \arrow[r, "(f_1,\dots,f_n)"] & \mathcal{O}_Y \arrow[r] & {}
\end{tikzcd}$$

The top complex is exact. Hence the induced morphism

$$\mathcal{O}_Y(-nE) \longrightarrow \pi^* \operatorname{Kos}(\underline{f};R)$$

is zero in the derived category $D(Y)$, since it factors through an exact complex. Applying $\mathbf{R}\pi_*(-)$ gives a morphism

$$\pi_*\mathcal{O}_Y(-nE) \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y(-nE) \longrightarrow \mathbf{R}\pi_*\pi^* \operatorname{Kos}(\underline{f};R).$$

By the projection formula,

$$\mathbf{R}\pi_*\pi^* \operatorname{Kos}(\underline{f};R) \simeq \operatorname{Kos}\big(\underline{f};\mathbf{R}\pi_*\mathcal{O}_Y\big).$$

Therefore the induced map

$$\overline{I^n} = \pi_*\mathcal{O}_Y(-nE) \longrightarrow \operatorname{Kos}\big(\underline{f};\mathbf{R}\pi_*\mathcal{O}_Y\big)$$

is zero in $D(R)$. By Kovács's lemma, the natural map

$$R \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y$$

splits in $D(R)$. Consequently, the induced map of Koszul complexes

$$\mathrm{Kos}(\underline{f}; R) \longrightarrow \mathrm{Kos}\big(\underline{f}; \mathbf{R}\pi_*\mathcal{O}_Y\big)$$

also splits in $D(R)$. We obtain the following commutative diagram in the derived category:

$$\begin{tikzcd}
\overline{I^n} = \pi_*\mathcal{O}_Y(-nE) \arrow[r, "0"] \arrow[d] & \mathrm{Kos}\big(\underline{f}; \mathbf{R}\pi_*\mathcal{O}_Y\big) \\
R = \pi_*\mathcal{O}_Y \arrow[r] & \mathrm{Kos}(\underline{f}; R) \arrow[u, "\text{split}"']
\end{tikzcd}$$

The zero map on the top row forces the composition

$$\overline{I^n} \longrightarrow R \longrightarrow \mathrm{Kos}(\underline{f}; R)$$

to be zero in $D(R)$. Now take zeroth cohomology. The zeroth cohomology of the Koszul complex $\mathrm{Kos}(\underline{f}; R)$ is

$$H^0\big(\mathrm{Kos}(\underline{f}; R)\big) \cong R/I.$$

Thus the induced map

$$\overline{I^n} \longrightarrow R \longrightarrow R/I$$

is zero. This says precisely that every element of $\overline{I^n}$ maps to zero in R/I, and hence

$$\overline{I^n} \subseteq I.$$

This proves the case $k = 1$. □

Remark 1.20. The proof for general $k \geq 1$ is obtained by replacing the last Koszul target with the appropriate higher Koszul filtration term. Conceptually, the same normalized blowup argument controls the map from $\pi_*\mathcal{O}_Y(-(n+k-1)E) = \overline{I^{n+k-1}}$ into the Koszul complex and shows that its image in R/I^k is zero. Therefore one obtains

$$\overline{I^{n+k-1}} \subseteq I^k.$$

The lecture notes above explicitly wrote out the $k = 1$ case, which already contains the essential derived splitting mechanism.

Summary of the Mechanism

The pseudo-rational Briançon–Skoda argument uses the following chain of ideas:

(i) Replace the ideal I by its normalized blowup

$$\pi\colon Y \to \operatorname{Spec}(R),$$

where $I\mathcal{O}_Y = \mathcal{O}_Y(-E)$ becomes invertible.

(ii) Interpret integral closures of powers of I geometrically:

$$\overline{I^n} = \pi_*\mathcal{O}_Y(-nE).$$

(iii) Use the Koszul complex on generators $f_1, \dots, f_n$ of I.

(iv) On Y, the twisted Koszul complex involving $\mathcal{O}_Y(-jE)$ is exact.

(v) Therefore the relevant map from $\mathcal{O}_Y(-nE)$ into the pulled-back Koszul complex is zero in the derived category.

(vi) Push forward to $\operatorname{Spec}(R)$.

(vii) Use the derived splitting

$$R \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y$$

supplied by pseudo-rationality.

(viii) Conclude that

$$\overline{I^n} \longrightarrow R/I$$

is zero, equivalently

$$\overline{I^n} \subseteq I.$$

2 A Gorenstein criterion in positive characteristics via the F-pure threshold by Suchitra Pande

Throughout, unless otherwise stated, $(R, \mathfrak{m}, k)$ denotes a Noetherian local ring. When working in positive characteristic, we assume $\operatorname{char}(R) = p > 0$.

Cohen–Macaulay and Gorenstein Rings

Definition 2.1 (Cohen–Macaulay local ring)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. We say that R is *Cohen–Macaulay* if

$$\operatorname{depth} R = d.$$

Equivalently, some, hence every, system of parameters

$$x_1, \ldots, x_d \in \mathfrak{m}$$

is an R-regular sequence.

Definition 2.2 (Gorenstein local ring)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. We say that R is *Gorenstein* if R is Cohen–Macaulay and, for some, equivalently every, system of parameters $x_1, \ldots, x_d$, the Artinian quotient

$$A = R/(x_1, \ldots, x_d)$$

has one-dimensional socle:

$$\dim_k \operatorname{Soc}(A) = \dim_k (0 :_A \mathfrak{m}).$$

Equivalently, if R admits a canonical module ω_R, then R is Gorenstein if and only if

$$\omega_R \cong R.$$

Question 2.3. What are effective criteria for deciding whether a given ring is Gorenstein?

Remark 2.4 (Characteristic $p > 0$ viewpoint). The answers discussed in this lecture come from F-singularity theory in prime characteristic.

The Frobenius Map and F-Finite Rings

Let

$$F^e : R \longrightarrow R, \qquad r \longmapsto r^{p^e},$$

be the e-fold Frobenius ring homomorphism. The target can be viewed as a new R-module, denoted $F^e_* R$. As an abelian group, $F^e_* R = R$, but the R-module structure is twisted by Frobenius:

$$s \cdot F^e_* r = F^e_* (s^{p^e} r), \qquad s, r \in R.$$

Thus the Frobenius maps give a sequence of R-modules

$$F_* R, \; F^2_* R, \; F^3_* R, \ldots .$$

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[rr, bend left, "F^2"] \arrow[rrr, bend left, "F^3"] & F_*R \arrow[r, "F"] & F_*^2R \arrow[r, "F"] & F_*^3R \end{tikzcd}$$

Definition 2.5 (F-finite)**.** A ring R of characteristic $p > 0$ is called *F-finite* if F_*R is a finite R-module. Equivalently, F_*^eR is finite over R for every $e \geq 1$.

Remark 2.6 (Standing assumption in the lecture)*.* The notes assume that $(R, \mathfrak{m})$ is a domain and is finite under Frobenius; that is, R is F-finite.

Regularity and F-Regularity

Theorem 2.7 (Kunz)**.** *Let R be a Noetherian local ring of characteristic $p > 0$. Then R is regular if and only if F_*^eR is free as an R-module for some, equivalently every, $e \geq 1$.*

Definition 2.8 (Strong F-regularity; Hochster–Huneke)**.** Let R be an F-finite reduced ring of characteristic $p > 0$. We say that R is *strongly F-regular* if for every element

$$c \in R$$

not contained in any minimal prime of R, there exists $e \gg 0$ such that the R-linear map

$$R \longrightarrow F_*^eR, \qquad 1 \longmapsto F_*^ec,$$

splits as a map of R-modules.

$$\begin{tikzcd} R \arrow[r, "1 \mapsto F_*^ec"] \arrow[rr, bend right, "\mathrm{id}_R"'] & F_*^eR \arrow[r, dashed, "\exists\ \psi"] & R \end{tikzcd}$$

Theorem 2.9 (Hochster–Huneke)**.** *If R is strongly F-regular, then R is normal and Cohen–Macaulay.*

Criterion 2.10 (Fedder's criterion)**.** *Fedder's criterion provides an effective test for F-purity and strong F-regularity for quotient rings $R = S/I$, especially when S is a* *regular local ring or a polynomial ring over a field.*

Remark 2.11. The abbreviation "SFR" refers to *strong F-regularity.* Fedder's criterion is often used to verify strong F-regularity by translating Frobenius splitting conditions into colon-ideal conditions in an ambient regular ring.

F-Pure Thresholds

Definition

Definition 2.12 (F-pure threshold)**.** Let $(R, \mathfrak{m})$ be an F-finite local domain of characteristic $p > 0$, and assume that R is F-pure. For each $e \geq 1$, define

$$\nu_e = \max\left\{ n \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \text{there exists } 0 \neq f \in \mathfrak{m}^n \text{ such that} \\ R \longrightarrow F_*^e R, \quad 1 \longmapsto F_*^e f, \text{ splits} \end{array} \right\}.$$

Following Takagi–Watanabe, the *F-pure threshold* of $(R, \mathfrak{m})$ is

$$\operatorname{fpt}(R) = \lim_{e \to \infty} \frac{\nu_e}{p^e}.$$

Remark 2.13 (Duplicate formulation from the lecture notes). The lecture notes also stated the same definition in the following shorthand form:

$$\nu_e = \max\left\{ n \;| 0 \neq f \in \mathfrak{m}^n \text{ such that } R \to F_*^e R \text{ splits with } 1 \mapsto F_*^e f \right\},$$

and

$$\operatorname{fpt}(R) = \lim_{e \to \infty} \frac{\nu_e}{p^e}.$$

Remark 2.14. The number ν_e measures how deep inside the maximal ideal one can choose an element f while still preserving a Frobenius splitting after twisting by f. The asymptotic ratio ν_e / p^e is the F-pure threshold.

A Determinantal Example

Example 2.15 (Determinantal rings)**.** Let

$$X = (x_{ij})_{m \times n}$$

be an $m \times n$ matrix of variables over a field k, where

$$2 \le r \le m \le n.$$

Let I_r denote the ideal of $r \times r$ minors of X, and set

$$R = k[X]/I_r.$$

Then:

- R is strongly F-regular, hence R is Cohen–Macaulay.
- R is Gorenstein if and only if $m = n$.
- The lecture notes recorded the numerical invariants

$$\operatorname{fpt}(R) = m(r-1), \qquad a(R) = -n(r-1).$$

Remark 2.16. In this example, equality

$$\operatorname{fpt}(R) = -a(R)$$

is equivalent to

$$m(r-1) = n(r-1),$$

and since $r \ge 2$, this is equivalent to $m = n$. Thus the determinantal example illustrates the slogan:

$$\text{Gorenstein} \quad \Longleftrightarrow \quad \operatorname{fpt}(R) = -a(R).$$

The Gorenstein Criterion

The Hirose–Watanabe–Yoshida Conjecture

Conjecture 2.17 (Hirose–Watanabe–Yoshida)**.** *Let S be a standard graded algebra over a field k. Let $\mathfrak{m}$ be the homogeneous maximal ideal, and set*

$$R = S_{\mathfrak{m}}.$$

Assume that S is strongly F-regular. Then:

(i)

$$\operatorname{fpt}(R) \le -a(R).$$

(ii)
$$R \textit{ is Gorenstein} \iff \operatorname{fpt}(R) = -a(R).$$

Here the a-invariant is given by

$$a(R) = \max\left\{n \in \mathbb{Z} \,\middle|\, [H^d_{\mathfrak{m}}(S)]_n \neq 0\right\},$$

where $d = \dim S$.

Previous Results

Remark 2.18 (Previous results). We have the following known cases and partial results:

(i) Hirose–Watanabe–Yoshida proved the toric case.

(ii) Chiba–Matsuda proved the Hibi ring case.

(iii) De Stefani–Núñez-Betancourt proved:

  (i)
$$\operatorname{fpt}(R) \leq -a(R);$$

  (ii) the "only if" direction:
$$R \text{ Gorenstein} \implies \operatorname{fpt}(R) = -a(R).$$

(iv) Singh–Takagi–Varbaro proved the converse direction
$$\operatorname{fpt}(R) = -a(R) \implies R \text{ is Gorenstein}$$
under the additional assumption that the anti-canonical ring is finitely generated.

Main Theorem

Theorem 2.19 (Main theorem)**.** *Let S be a standard graded, strongly F-regular ring over an F-finite field of characteristic $p > 0$. Let $\mathfrak{m}$ be the homogeneous maximal ideal. Then*
$$\operatorname{fpt}_{\mathfrak{m}}(S) \leq -a(S),$$
and
$$S \textit{ is Gorenstein} \iff \operatorname{fpt}_{\mathfrak{m}}(S) = -a(S).$$

Geometric Setup

Let

$$X = \mathrm{Proj}(S).$$

Write

$$\mathcal{O}_X(1) = \mathcal{O}_X(L)$$

for the very ample line bundle associated to the standard grading. Thus L is an ample Cartier divisor on X. When the lecture notes use Y, we identify it with the same projective scheme:

$$Y = \mathrm{Proj}(S) = X.$$

Remark 2.20. For a standard graded k-algebra S, information about the graded local cohomology module $H^\bullet_{\mathfrak{m}}(S)$ is translated into sheaf cohomology on $X = \mathrm{Proj}(S)$. This is the bridge between the algebraic invariant $a(S)$ and the geometry of K_X.

A Divisor Lemma

Lemma 2.21. *Let Y be a projective variety, let L be an ample divisor on Y, and let $D \geq 0$ be an effective Weil divisor. Suppose that for each e there exist effective $\mathbb{Q}$-divisors*

$$\Delta_e \geq 0$$

and positive rational numbers

$$\lambda_e \in \mathbb{Q}_{>0}$$

such that

$$D + \Delta_e \sim_{\mathbb{Q}} \lambda_e L$$

and

$$\lim_{e\to\infty} \lambda_e = 0.$$

Then

$$D = 0.$$

Proof. Assume $D \neq 0$. Since L is ample, one can choose an integer $t \gg 0$ and a curve $C \subseteq Y$ obtained as a sufficiently general complete intersection of members of $|tL|$ such that C is not contained in the support of $D + \Delta_e$, and $D \cdot C > 0$. Since $\Delta_e \geq 0$, we have

$$(D + \Delta_e) \cdot C \geq D \cdot C > 0.$$

On the other hand,

$$D + \Delta_e \sim_{\mathbb{Q}} \lambda_e L$$

implies

$$(D + \Delta_e) \cdot C = \lambda_e (L \cdot C).$$

Since $\lambda_e \to 0$, the right-hand side tends to 0, contradicting the fixed strict positivity $D \cdot C > 0$. Hence $D = 0$. □

Remark 2.22. The intuitive idea is that a fixed nonzero effective divisor cannot be $\mathbb{Q}$-linearly equivalent, even after adding more effective divisor, to arbitrarily small positive multiples of an ample divisor.

Proof Sketch

Proof. Let

$$Y = \mathrm{Proj}(S)$$

be a projective variety over $k = \overline{k}$. Let

$$\mathcal{L} = \mathcal{O}_Y(1)$$

be an ample line bundle, and let ω_Y denote the canonical sheaf of Y.

Step 1: The a-invariant produces an effective divisor. The lecture notes recorded the following chain:

$$[H^d_{\mathfrak{m}}(S)]_n \cong H^d(Y, \mathcal{L}^n) \cong_{\text{Serre}} H^0(Y, \omega_Y \otimes \mathcal{L}^{-n}).$$

Thus, for $n \in \mathbb{Z}$,

$$a = -a(S) = \min \left\{ n \,\middle|\, H^0(Y, \omega_Y \otimes \mathcal{L}^n) \neq 0 \right\}.$$

Therefore there exists an effective divisor $D \geq 0$ such that

$$D \sim K_Y + aL.$$

Step 2: Frobenius splittings become sections of anti-canonical twists. The lecture notes express the key correspondence as

$$\left\{ \begin{array}{c} \text{homogeneous splittings} \\ \phi : F^e_* S \to S \\ \text{taking } f_n \in S_n \text{ to } 1 \end{array} \right\} \quad \Longleftrightarrow \quad \mathrm{Hom}_{\mathcal{O}_Y}\left(F^e_* \mathcal{L}^n, \mathcal{O}_Y\right).$$

By duality for Frobenius,

$$\mathrm{Hom}_{\mathcal{O}_Y}\left(F^e_* \mathcal{L}^n, \mathcal{O}_Y\right) \cong F^e_* \left(\omega_Y^{1-p^e} \otimes \mathcal{L}^{-n} \right).$$

Therefore a splitting ϕ mapping $F^e_*(f_n)$ to 1, with $f_n \in S_n$, yields

$$H^0\left(Y, F^e_*(\omega_Y^{1-p^e} \otimes \mathcal{L}^{-n})\right) \cong H^0\left(Y, \omega_Y^{1-p^e} \otimes \mathcal{L}^{-n}\right).$$

Equivalently, one obtains an effective divisor

$$0 \leq D_\phi \sim (1-p^e)K_Y - nL.$$

Define

$$\Delta_\phi = \frac{1}{p^e-1} D_\phi.$$

Then

$$\Delta_\phi \sim_{\mathbb{Q}} -K_Y - \frac{n}{p^e-1} L.$$

$$\phi : F^e_* S \to S,\ \phi(F^e_* f_n) = 1 \rightsquigarrow 0 \neq s_\phi \in H^0\left(Y, \omega_Y^{1-p^e} \otimes \mathcal{L}^{-n}\right) \rightsquigarrow 0 \leq D_\phi \sim (1-p^e)K_Y -$$

The guiding idea is:

$$F\text{-splittings give sections of } -K_Y \text{ up to twists,}$$

whereas the a-invariant gives sections of K_Y up to twists.

Step 3: Apply the definition of ν_e. For each e, choose

$$f_e \in S_{\nu_e}$$

and a splitting

$$\phi_e : F^e_* S \to S$$

such that

$$\phi_e(F^e_* f_e) = 1.$$

Then the preceding discussion gives

$$0 \leq \Delta_{\phi_e} \sim_{\mathbb{Q}} -K_Y - \frac{\nu_e}{p^e-1} L.$$

Combining this with

$$D \sim K_Y + aL$$

gives

$$D + \Delta_{\phi_e} \sim_{\mathbb{Q}} \left(a - \frac{\nu_e}{p^e-1}\right) L.$$

Step 4: Derive the inequality and the Gorenstein criterion. On a projective variety, any effective divisor has nonnegative degree against ample curves. Therefore

$$a \geq \frac{\nu_e}{p^e - 1}.$$

Passing to the limit gives

$$a \geq \operatorname{fpt}(S).$$

Equivalently,

$$\operatorname{fpt}(S) \leq -a(S).$$

If equality holds in the conjecture, namely if

$$\operatorname{fpt}(R) = a = -a(S),$$

then

$$\lambda_e = a - \frac{\nu_e}{p^e - 1} \longrightarrow 0.$$

By Lemma 2.21,

$$D = 0.$$

Therefore

$$0 = D \sim K_Y + aL,$$

so

$$K_Y = -aL.$$

Consequently

$$\omega_S \cong S(-a).$$

Hence ω_S is a free rank-one graded S-module, so S is Gorenstein. $\square$

Corrected and Polished Proof Sketch

Corrected proof sketch. Let

$$X = \operatorname{Proj}(S),$$

and let L be the ample divisor corresponding to $\mathcal{O}_X(1)$. Put

$$a := -a(S).$$

Assume, for notational clarity, that

$$\dim S = d + 1, \qquad \dim X = d.$$

The standard comparison between graded local cohomology and sheaf cohomology gives

$$H^{d+1}_{\mathfrak{m}}(S) \cong \bigoplus_{r \in \mathbb{Z}} H^d(X, \mathcal{O}_X(rL)).$$

Hence

$$a = \min \left\{ r \in \mathbb{Z} \,\middle|\, H^d(X, \mathcal{O}_X(-rL)) \neq 0 \right\}.$$

By Serre duality, this is equivalent to

$$a = \min \left\{ r \in \mathbb{Z} \,\middle|\, H^0(X, \mathcal{O}_X(K_X + rL)) \neq 0 \right\}.$$

Therefore there exists an effective Weil divisor $D \geq 0$ such that

$$D \sim K_X + aL.$$

Now let $f_e \in S_{n_e}$ be homogeneous, and suppose there exists a splitting

$$\phi_e : F^e_* S \to S$$

such that

$$\phi_e(F^e_* f_e) = 1.$$

Using duality for Frobenius on X, such a splitting gives an effective Weil divisor $D_e \geq 0$ satisfying

$$\frac{1}{p^e - 1} D_e \sim_{\mathbb{Q}} -K_X - \frac{n_e}{p^e - 1} L.$$

Adding this to

$$D \sim K_X + aL$$

yields

$$D + \frac{1}{p^e - 1} D_e \sim_{\mathbb{Q}} \left(a - \frac{n_e}{p^e - 1} \right) L.$$

Since the left-hand side is effective, its intersection against sufficiently general complete intersections of ample divisors is nonnegative. Hence

$$\frac{n_e}{p^e - 1} \leq a.$$

Taking $n_e = \nu_e$ and passing to the limit gives

$$\operatorname{fpt}_{\mathfrak{m}}(S) \leq a = -a(S).$$

Now suppose equality holds:

$$\operatorname{fpt}_{\mathfrak{m}}(S) = a.$$

Then one can choose $n_e = \nu_e$ so that

$$a - \frac{\nu_e}{p^e - 1} \longrightarrow 0.$$

Applying Lemma 2.21 to

$$D + \frac{1}{p^e - 1} D_e \sim_{\mathbb{Q}} \left(a - \frac{\nu_e}{p^e - 1} \right) L$$

forces

$$D = 0.$$

Thus

$$K_X \sim -aL.$$

Consequently the graded canonical module is isomorphic to a shift:

$$\omega_S \cong S(-a).$$

Hence S is quasi-Gorenstein. Since S is strongly F-regular, it is Cohen–Macaulay. Therefore quasi-Gorensteinness implies that S is Gorenstein. Conversely, if S is Gorenstein, then the known direction of the Hirose–Watanabe–Yoshida criterion gives

$$\mathrm{fpt}_{\mathfrak{m}}(S) = -a(S).$$

Therefore

$$S \text{ is Gorenstein} \quad \Longleftrightarrow \quad \mathrm{fpt}_{\mathfrak{m}}(S) = -a(S).$$

□

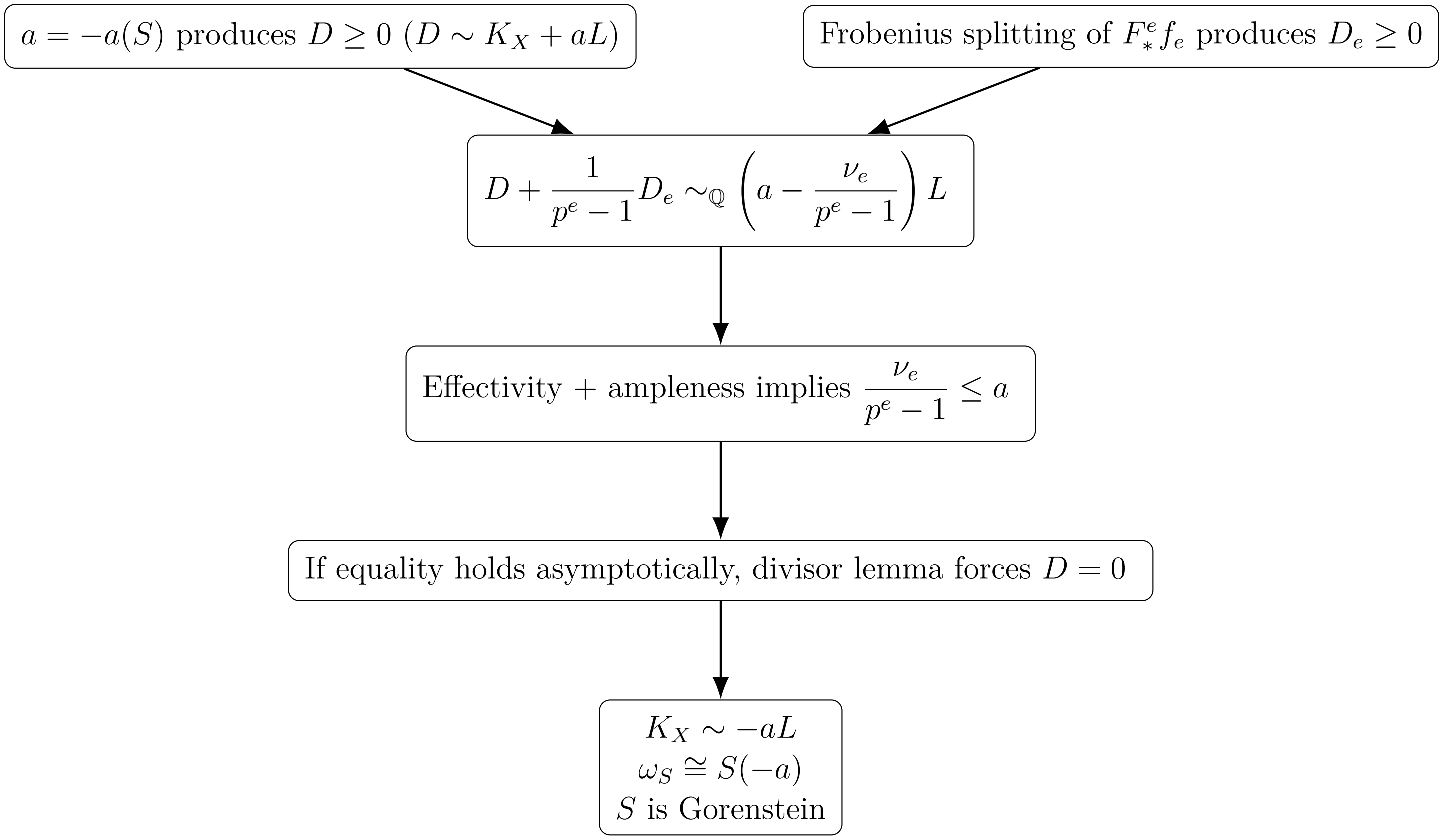
Remark 2.23. A standard graded ring S is quasi-Gorenstein if its canonical module is a graded shift of itself:

$$\omega_S \cong S(t)$$

for some $t \in \mathbb{Z}$. If S is also Cohen–Macaulay, then quasi-Gorensteinness is equivalent to being Gorenstein. Strong F-regularity supplies the Cohen–Macaulay property.

Summary of the Criterion

The geometric mechanism of the proof is the following.



Thus the proof compares two kinds of sections:

$$
\begin{aligned}
a\text{-invariant} \quad &\rightsquigarrow \quad \text{sections of } K_X + aL, \\
F\text{-pure threshold} \quad &\rightsquigarrow \quad \text{sections of } -K_X - \frac{\nu_e}{p^e - 1} L.
\end{aligned}
$$

Adding these divisor classes produces an effective divisor linearly equivalent to

$$\left(a - \frac{\nu_e}{p^e - 1}\right) L.$$

This gives the inequality

$$\operatorname{fpt}_{\mathfrak{m}}(S) \leq -a(S),$$

and equality forces the canonical class to be an integral multiple of the hyperplane class, yielding the Gorenstein conclusion.

3 Koszul Duality for Polynomial Rings by Michael Brown

Let k be a field, and let A be an abelian group. Let

$$S = k[x_0, \ldots, x_n]$$

be a positively A-graded polynomial ring. Thus each variable x_i is assigned a degree

$$\deg(x_i) \in A,$$

and there exists a group homomorphism

$$\theta : A \longrightarrow \mathbb{Z}$$

such that, for every homogeneous element $s \in S$,

$$\theta(\deg(s)) \geq 0,$$

and

$$\theta(\deg(s)) = 0 \quad \Longleftrightarrow \quad s \in k.$$

Equivalently, the induced $\mathbb{Z}$-grading coming from θ is nonnegative, and the only homogeneous elements of θ-degree zero are the scalar elements of k. Let

$$E = \bigwedge_k (e_0, \ldots, e_n)$$

be the exterior algebra over k on generators $e_0, \ldots, e_n$. We give E an $(A \oplus \mathbb{Z})$-grading by declaring

$$\deg(e_i) = \Big(-\deg(x_i), -1\Big).$$

Here the first component records the internal A-degree, while the second component records the exterior or homological degree.

Definition 3.1 (Differential E-module)**.** A *differential E-module* is an $(A \oplus \mathbb{Z})$-graded E-module D together with an endomorphism

$$\partial : D \longrightarrow D$$

of degree $(0, -1)$ such that

$$\partial^2 = 0.$$

Remark 3.2. The condition that ∂ has degree $(0, -1)$ means that ∂ preserves the internal A-degree and lowers the exterior or homological degree by one. This is precisely the grading behavior of the usual Koszul differential, where multiplication by x_i is paired with exterior multiplication by e_i.

Let

$$\mathrm{Com}(S)$$

denote the category of complexes of graded S-modules, and let

$$\mathrm{DM}(E)$$

denote the category of differential E-modules.

The Functor from Graded S-Modules to Differential E-Modules

For a graded S-module M, define

$$F(M) = \bigoplus_{a \in A} M_a \otimes_k E.$$

The differential on $F(M)$ is given by

$$\partial_{F(M)}(m \otimes e) = \sum_{i=0}^{n} x_i m \otimes e_i e,$$

for homogeneous $m \in M$ and $e \in E$.

Lemma 3.3. *The map $\partial_{F(M)}$ is a differential of degree $(0, -1)$; that is,*

$$\partial^2_{F(M)} = 0.$$

Proof. First observe that $x_i m$ has A-degree $\deg(m) + \deg(x_i)$, while multiplication by e_i has degree $(-\deg(x_i), -1)$. Hence each summand

$$x_i m \otimes e_i e$$

has the same total A-degree as $m \otimes e$ and exterior degree one less. Thus the differential has degree $(0, -1)$. Now compute:

$$\begin{aligned}
\partial^2_{F(M)}(m \otimes e) &= \partial_{F(M)}\left(\sum_{i=0}^{n} x_i m \otimes e_i e\right) \\
&= \sum_{i,j=0}^{n} x_j x_i m \otimes e_j e_i e.
\end{aligned}$$

Since the variables x_i commute in S and the generators e_i anticommute in the exterior algebra, the terms with $i \neq j$ cancel in pairs:

$$x_j x_i m \otimes e_j e_i e + x_i x_j m \otimes e_i e_j e = x_i x_j m \otimes (e_j e_i + e_i e_j) e = 0.$$

The terms with $i = j$ vanish because $e_i^2 = 0$. Therefore $\partial_{F(M)}^2 = 0$. □

If

$$M^\bullet \in \mathrm{Com}(S)$$

is a complex of graded S-modules with differential d_M, then the correct total differential on $F(M^\bullet)$ is

$$\partial(m \otimes e) = d_M(m) \otimes e + \sum_{i=0}^{n} x_i m \otimes e_i e,$$

up to the usual sign convention determined by the homological grading.

Remark 3.4. More explicitly, if m is homogeneous of homological degree p, then one common convention is

$$\partial(m \otimes e) = d_M(m) \otimes e + (-1)^p \sum_{i=0}^{n} x_i m \otimes e_i e.$$

The sign guarantees that the internal differential d_M and the Koszul-type differential anticommute, so that the total differential squares to zero.

Example 3.5. The residue field $k = S/(x_0, \dots, x_n)$ satisfies

$$F(k) = E.$$

Indeed, multiplication by every x_i is zero on k, so the induced differential on $F(k)$ is zero.

Example 3.6. Let

$$S = k[x_0, x_1], \qquad \deg(x_0) = 1, \qquad \deg(x_1) = 2.$$

Then

$$E = \bigwedge_k (e_0, e_1), \qquad \deg(e_0) = (-1, -1), \qquad \deg(e_1) = (-2, -1).$$

The functor F applied to S gives a differential E-module whose underlying graded pieces may be visualized as follows:

$$\begin{tikzcd}
E \arrow[r, "e_0"] \arrow[rr, bend right, "e_1"'] & E(-1,0) \arrow[r, "e_0"] \arrow[rr, bend right, "e_1"'] & E(-2,0) \arrow[r, "e_0"] \arrow[rr, bend right, "e_1"'] & E(-3,0) \arrow[r, "e_0"] & \cdots \\
1 \arrow[r, mapsto] & x_0 \arrow[r, mapsto] & \begin{matrix} x_0^2 \\ x_1 \end{matrix} \arrow[r, mapsto] & \begin{matrix} x_0^3 \\ x_0x_1 \end{matrix} \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

The straight arrows represent the contribution of multiplication by x_0, paired with exterior multiplication by e_0. The curved arrows represent the contribution of multiplication by x_1, paired with exterior multiplication by e_1.

Remark 3.7. The notation $E(-a, 0)$ denotes a shift in the internal degree. The diagram records the fact that the degree-d component of S is spanned by all monomials $x_0^u x_1^v$ with

$$u + 2v = d.$$

For instance, in degree 2, the monomials are x_0^2 and x_1.

The Homotopy Category of Free Differential E-Modules

Let

$$K_{\mathrm{DM}}(E) = \{\text{free differential } E\text{-modules}\} / \text{homotopy}.$$

Thus objects are free differential E-modules, and morphisms are taken modulo homotopy.

Theorem 3.8 (Koszul duality for positively graded polynomial rings)**.** *The functor F induces an equivalence*

$$F : \mathrm{D}(S) \xrightarrow{\sim} K_{\mathrm{DM}}(E).$$

Furthermore, after restricting to bounded complexes, one obtains an equivalence

$$F : \mathrm{D}^{\mathrm{b}}(S) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}{}_{\mathrm{DM}}(E).$$

Remark 3.9. This is a form of Koszul duality between the symmetric algebra S and the exterior algebra E. The functor F packages the Koszul differential into the structure of a single differential E-module.

Remark 3.10. One should not confuse the preceding equivalence with an equivalence

$$F : \mathrm{D}(S) \xrightarrow{\sim} \mathrm{D}_{\mathrm{DM}}(E).$$

In general, F does not identify the full derived category of graded S-modules with the ordinary derived category of all differential E-modules. The target must be interpreted as the appropriate homotopy category of free differential E-modules.

Example 3.11. When $n = 0$, one has $S = k[x]$. In this case,

$$F(S[x^{-1}])$$

is exact.

Remark 3.12. The preceding example illustrates why the precise target category matters. Exactness in the category of differential E-modules does not automatically imply triviality in the relevant homotopy category. Thus the correct statement is naturally formulated using $K_{\mathrm{DM}}(E)$ rather than the full derived category of differential E-modules.

Koszul Duality for Toric Varieties

Let X be a smooth projective toric variety. By the Cox construction, X corresponds to the following algebraic data:

$$X \quad \longleftrightarrow \quad \begin{cases} \text{a Cox ring } S = k[x_0, \dots, x_n] \text{ graded by } \operatorname{Pic}(X), \\ \text{an irrelevant ideal } B \subseteq S, \text{ which is square-free monomial.} \end{cases}$$

The grading by $\operatorname{Pic}(X)$ records the divisor class of each torus-invariant divisor.

Example 3.13 (Products of projective spaces)**.** Let

$$X = \mathbb{P}^m \times \mathbb{P}^n.$$

Then

$$\operatorname{Pic}(X) \cong \mathbb{Z}^2.$$

The Cox ring is

$$S = k[x_0, \dots, x_m, y_0, \dots, y_n],$$

with grading

$$\deg(x_i) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \deg(y_j) = \begin{pmatrix}0\\1\end{pmatrix}.$$

The irrelevant ideal is

$$B = (x_0, \dots, x_m) \cap (y_0, \dots, y_n).$$

Example 3.14 (Hirzebruch surfaces)**.** Let

$$X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)\right), \qquad r \geq 0.$$

Then

$$\mathrm{Pic}(X) \cong \mathbb{Z}^2.$$

The Cox ring is

$$S = k[x_0, x_1, y_0, y_1].$$

The grading is given by

$$\deg(x_i) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \deg(y_0) = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \deg(y_1) = \begin{pmatrix}-r\\1\end{pmatrix}.$$

The irrelevant ideal is

$$B = (x_0, x_1) \cap (y_0, y_1).$$

Let H be an ample line bundle on X. This induces a homomorphism

$$\theta : \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

given by

$$\theta(D) = \left\langle D, H^{\dim X - 1} \right\rangle.$$

Here the bracket denotes the intersection product. Since H is ample, this homomorphism provides a positivity function on $\mathrm{Pic}(X)$ suitable for the positively graded setup.

Remark 3.15. For toric varieties, the Cox ring is typically multigraded by $\mathrm{Pic}(X)$ rather than singly graded by $\mathbb{Z}$. The ample line bundle H converts this multigrading into a numerical positivity condition by intersecting divisor classes against a high power of H.

The Classical BGG Correspondence

Let

$$E = \bigwedge_k (e_0, \dots, e_n)$$

with

$$\deg(e_i) = -1.$$

Theorem 3.16 (Bernstein–Gelfand–Gelfand; Buchweitz)**.** *There is an equivalence*

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n) \simeq K^{\mathrm{ex}}(E),$$

where

$$K^{\mathrm{ex}}(E) = \{\textit{exact complexes of finitely generated graded free } E\textit{-modules}\}.$$

Moreover, the singularity category of E is

$$\mathrm{D}_{\mathrm{sg}}(E) = \mathrm{D}^{\mathrm{b}}(E)/\operatorname{Perf}(E).$$

Remark 3.17. The Bernstein–Gelfand–Gelfand correspondence relates coherent sheaves on projective space to algebraic data over the exterior algebra. A useful way to view the projective-space case is

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n) \simeq \mathrm{D}^{\mathrm{b}}(S)/\,\mathrm{D}^{\mathrm{b}}{}_{\mathfrak{m}}(S),$$

where

$$\mathfrak{m} = (x_0, \ldots, x_n)$$

is the irrelevant maximal ideal and $\mathrm{D}^{\mathrm{b}}{}_{\mathfrak{m}}(S)$ denotes the full subcategory of complexes with homology supported at $\mathfrak{m}$.

The work of Eisenbud–Fløystad–Schreyer gives an algorithmic form of this correspondence, especially for computing sheaf cohomology over projective space. The goal is to extend this perspective from projective space to a general smooth projective toric variety X. For a smooth projective toric variety X, set

$$E = \bigwedge_k (e_0, \ldots, e_n),$$

now regarded as a $(\operatorname{Pic}(X) \oplus \mathbb{Z})$-graded exterior algebra with

$$\deg(e_i) = \big(-\deg(x_i), -1\big).$$

The desired comparison is

$$\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(S)/\,\mathrm{D}^{\mathrm{b}}{}_{B}(S) \simeq \mathrm{D}^{\mathrm{b}}{}_{\mathrm{DM}}(E)/\boxed{?}.$$

Here $\mathrm{D}^{\mathrm{b}}{}_{B}(S)$ denotes the full subcategory of $\mathrm{D}^{\mathrm{b}}(S)$ consisting of complexes whose homology is supported on the irrelevant locus defined by B.

Remark 3.18. The box indicates the main structural issue in extending BGG from projective space to toric varieties: one must identify the correct subcategory of differential E-modules that corresponds to B-torsion on the Cox-ring side.

Irrelevant Subsets and Quotients of the Exterior Algebra

Definition 3.19 (Irrelevant subset)**.** A subset

$$I \subseteq \{0, \dots, n\}$$

is called *irrelevant* if

$$B \subseteq (x_i : i \in I).$$

Definition 3.20 (The ideal Θ_I)**.** For any subset

$$I \subseteq \{0, \dots, n\},$$

define the ideal

$$\Theta_I \subseteq E$$

by

$$\Theta_I = (e_i \in E : i \notin I).$$

Equivalently,

$$E/\Theta_I \cong \bigwedge_k (e_i : i \in I).$$

Example 3.21. If

$$X = \mathbb{P}^m \times \mathbb{P}^n$$

and

$$B = (x_0, \dots, x_m) \cap (y_0, \dots, y_n),$$

then a subset I of the variables is irrelevant precisely when the coordinate ideal generated by variables in I contains B. For example, any set I containing all x-variables or all y-variables gives an ideal $(x_i : i \in I)$ containing one of the factors defining B and hence containing B.

Remark 3.22. The construction of Θ_I is dual to restricting the polynomial variables to the subset indexed by I. Killing the exterior generators e_i with $i \notin I$ leaves precisely the exterior algebra associated to the variables indexed by I.

We define the category of exact differential modules with respect to the irrelevant subsets as

$$K_{\mathrm{DM}}^{\mathrm{ex}}(E) = \{F \in K_{\mathrm{DM}}(E) \mid F \otimes_E E/\Theta_I \text{ is exact for every irrelevant } I\}.$$

Derived Global Sections and the Brown–Erman Theorem

Let

$$\mathbf{R}\Gamma_*$$

denote the total derived global sections functor

$$\mathbf{R}\Gamma_*(\mathcal{F}) = \bigoplus_{d \in \mathrm{Pic}(X)} \mathbf{R}\Gamma\Big(X, \mathcal{F}(d)\Big).$$

Thus $\mathbf{R}\Gamma_*(\mathcal{F})$ is a complex of $\mathrm{Pic}(X)$-graded S-modules.

Remark 3.23. The functor $\mathbf{R}\Gamma_*$ is the derived version of the Cox-ring section functor. It simultaneously records the cohomology of all twists of $\mathcal{F}$ by line bundles corresponding to classes in $\mathrm{Pic}(X)$.

Theorem 3.24 (Brown–Erman)**.** *Let X be a smooth projective toric variety with Cox ring*

$$S = k[x_0, \dots, x_n]$$

and irrelevant ideal B. Let

$$E = \bigwedge_k (e_0, \dots, e_n)$$

with the $(\mathrm{Pic}(X) \oplus \mathbb{Z})$-grading

$$\deg(e_i) = \Big(-\deg(x_i), -1\Big).$$

Then the composition

$$F \circ \mathbf{R}\Gamma_* : \mathrm{D}(X) \longrightarrow K_{\mathrm{DM}}^{\mathrm{ex}}(E)$$

is an equivalence of categories.

Remark 3.25. This theorem gives a toric analogue of the BGG correspondence. The category $K_{\mathrm{DM}}^{\mathrm{ex}}(E)$ replaces the category of exact complexes of free exterior modules appearing in the projective-space case. The additional exactness condition after tensoring with E/Θ_I encodes the irrelevant ideal B and therefore the toric geometry of X.

Summary of the Correspondence

The lecture can be summarized by the following diagram:

$$\begin{tikzcd}
\mathrm{D}(X) \arrow[r, "\mathbf{R}\Gamma_*"] \arrow[rr, bend left, "F\circ\mathbf{R}\Gamma_*"] & \mathrm{D}(S) \arrow[r, "F"] \arrow[d, two heads] & K_{\mathrm{DM}}(E) \arrow[d, two heads] \\
 & \mathrm{D}(S)/\,\mathrm{D}_B(S) \arrow[r, dashed, "\sim"] & K_{\mathrm{DM}}^{\mathrm{ex}}(E)
\end{tikzcd}$$

The quotient by $\mathrm{D}_B(S)$ on the Cox-ring side corresponds to imposing exactness conditions on the exterior-algebra side after tensoring with the quotients E/Θ_I for all irrelevant subsets I.

Remark 3.26. For $X = \mathbb{P}^n$, the irrelevant ideal is simply

$$B = (x_0, \ldots, x_n),$$

and the toric statement specializes to the classical BGG correspondence. For a general smooth projective toric variety, the irrelevant ideal is usually an intersection of monomial ideals, and this forces the more refined exactness condition defining $K_{\mathrm{DM}}^{\mathrm{ex}}(E)$.