URiCA 2025

A conference for upcoming researchers in Commutative Algebra

Lecture 1 — Perfect Ideals with Fixed Deviation & Type by Xianglong Ni

Let S be a regular local or graded ring with $S \supseteq \mathbb{Q}$. For example, $S = \mathbb{C}[x_1, \dots, x_n]$ or a localization (completion).

Definition 1. An ideal $I \subseteq S$ is **perfect** if $\mathbb{F}^*$ is acyclic, where $\mathbb{F}$ is a minimal free resolution of S/I.

Let $\mathbb{F}$ be the minimal free resolution:

$$\mathbb{F}\colon 0 \to F_c \to \cdots \to F_1 \to F_0 = S \to S/I \to 0$$

The dual of the minimal free resolution is:

$$0 \to F_0^* \to F_1^* \to \cdots \to F_c^* \to \operatorname{Ext}_S^c(S/I, S) \to 0$$

Let $c = \operatorname{pdim} S/I = \operatorname{grade} I = \operatorname{codim} I$. (Since I is perfect, S is Cohen-Macaulay). By the Auslander-Buchsbaum formula:

$$I \text{ perfect} \iff S/I \text{ is C.M.}$$

Define the Betti numbers β_i:

$$\beta_i = \beta_i(S/I) = \operatorname{rk} F_i = \dim_k(\operatorname{Tor}_i(S/I, k))$$

We define two invariants:

- **Deviation** $d = \beta_1 - c$
- **Type** $t = \beta_c$

Example 1.
- $d = 0$: "complete intersection"
- $t = 1$: "Gorenstein"
- $d = 1$: "almost complete intersection"

Question: Can we upper bound the "complexity" of perfect ideals? (Fixing c, d, t) e.g., bounds for β_i for $2 \leq i \leq c - 1$.

Example 2. For $c = 4, d = 2, t = 2$, the Betti sequence is $\beta = (\beta_0, \beta_1, \dots) = (1, 6, \beta_2, \beta_3, 2)$.

If $d = 0$, $\mathbb{F}$ is isomorphic to the Koszul complex. In this case, $t = 1$, and $\beta_i = \binom{c}{i}$.

Observation 1. $\sum(-1)^i\beta_i = 0$.

- For $c = 2$: $0 \to S^t \to S^{2+d} \to S \to 0$. We have $t = d + 1$.
- For $c = 3$: trivial case. $0 \to S^t \to S^{2+d+t} \to S^{3+d} \to S \to 0$.
- First interesting case, $c = 4, t = 1 \implies \mathbb{F}^* \cong \mathbb{F}$.

$$0 \to S^1 \to S^{\beta_3} \to S^{\beta_2} \to S^{4+d} \to S \to 0$$

  This implies $\beta_3 = 4 + d$ and $\beta_2 = 6 + 2d$.

For $c = 4, d = 1$, we have:

$$0 \to S^t \to S^{\beta_3} \to S^{\beta_2} \to S^5 \to S \to 0$$

Linkage

Let $I \subset S$, and let $(\alpha_1, \ldots, \alpha_c) \subset I$ (i.e., $\alpha_1, \ldots, \alpha_c \in I$ is a regular sequence). Define:

$$J = (\alpha_1, \ldots, \alpha_c) : I = \{x \in S : xI \subset (\alpha_1, \ldots, \alpha_c)\}$$

If $I = (\alpha_1, \ldots, \alpha_c) : J$, we say I and J are **directly linked**, denoted $I \sim J$.

Remark 1. I is in the linkage class of a complete intersection (licci) if there exists a sequence of links: $I \sim \cdots \sim$ c.i.

If I is perfect, then J is perfect is automatic.

Ferrand Mapping Cone

Let $\mathbb{F}$ and $\mathbb{K}$ be the following resolutions, with a map ϕ between them:

$$\begin{array}{llllllllll}
\mathbb{F}: & 0 & \longrightarrow & F_c & \longrightarrow \cdots \longrightarrow & F_1 & \longrightarrow & S & \longrightarrow & (S/I \to 0) \\
 & & & \downarrow \phi & & \downarrow & & \| & & \\
\mathbb{K}: & 0 & \longrightarrow & \wedge^c K & \longrightarrow \cdots \longrightarrow & K & \longrightarrow & S & & (\text{where } K = (\alpha_1, \ldots, \alpha_c))
\end{array}$$

Then $\mathrm{Cone}(\phi)^*$ resolves S/J:

$$S \leftarrow F_c^* \oplus K^* \leftarrow \cdots \leftarrow F_2^* \oplus K \leftarrow F_1^* \leftarrow 0$$

For $c = 4, d = 1 \implies 0 \to S^t \to S^{\beta_3} \to S^{\beta_2} \to S^5 \to S \to S/I \to 0$ ($I \sim J$ for some Gorenstein J). From the resolutions:

$$0 \to S^1 \to S^n \to S^{2n-2} \to S^n \to S \to (S/J)$$
$$0 \to S \to S^4 \to S^6 \to S^4 \to S \to S/(\alpha_1, \ldots, \alpha_4) \to 0$$

By the mapping cone construction:

$$\implies S \leftarrow S^5 \leftarrow S^{n+t} \leftarrow S^{2n+2} \leftarrow S^n \quad (\text{yielding } \beta_2 \leq t + 10)$$

Known Bounds and Maximums

- $c = 5, t = 1$: $(d, t) \neq (1, 1)$ by [Kunz].
- $c = 5, d = 2, t = 1$: [Lopez]: A licci ideal in the above parameter has β one of the following:
  - $\beta = (1, 7, 16, 16, 7, 1)$ (hypersurface of codim 3 of 5 gen)
  - $\beta = (1, 7, 21, 21, 7, 1)$ (Huneke-Ulrich Examples)

  There are no known non-licci examples of perfect ideals with $(d, t) = (2, 1)$.

Theorem 1 (Huneke-Vascancelos '96)**.** *For $(d, t) = (2, 1)$ & S/I is generically C.I. & unobstructed (e.g. licci), explicitly construct an exact sequence:*

$$S^{\binom{c+3}{2}} \to S^{c+2} \to S \to (S/I \to 0)$$

In particular, $\beta_2 = \beta_{c-2} \leq \binom{c+3}{2}$.

- $(c, d, t) = (6, 2, 1) \implies \beta = (1, 8, \leq 36, \leq 36, 8, 1)$.

  Fact (M2)**.** *There is an ideal attaining these maximums.*

- $(c, d, t) = (7, 2, 1) \implies \beta = (1, 9, \leq 45, -, -, 9, 1)$.
- $(c, d, t) = (4, 2, 2) \implies \beta = (1, 6, -, -, 2)$.

  For these two cases, there exist ideals with arbitrarily large intermediate β_i.

Proof Sketch.
1. Take a "suitable" homogeneous $I_0 \subsetneq \mathbb{C}[x_1, \ldots, x_n]$, $n \geq c$.
2. Define $I_j = (\alpha_1, \ldots, \alpha_c) : I_{j-1} \implies$ regular sequence of maximum degree among minimal generators of I_{j-1}.

This process yields $\lim_{N \to \infty} \beta_3(S/I_N) = \infty$. □

Herzog Classes

Let $S = \mathbb{C}[x_1, \ldots, x_n]$ $(n \geq c)$. Define P_c of ideals in S as follows:

- $(l_1, \ldots, l_c) \in P_c$, where $l_1, \ldots, l_c$ are independent linear forms.
- If $I \in P_c$ with minimal free resolution:

$$0 \to \bigoplus_{j=1}^{t} S(-b_{cj}) \to \cdots \to \bigoplus_{j=1}^{c+d} S(-b_{1j}) \to S \to S/I \to 0$$

- $\varkappa = \frac{\sum_{j=1}^{t}(b_{cj} - c + 2)}{t+1} = \frac{1 + \sum_{j=1}^{c+d}(b_{1j} - 1)}{d+1}$
- $\lambda_i = \varkappa + 1 - b_{1i}$
- $N_i = \varkappa + c - 2 - b_{ci}$

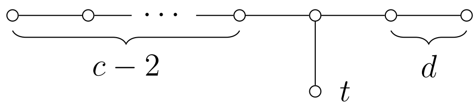
Suppose $I = (h_1, \ldots, h_N)$, $\deg h_i = \varkappa + 1 - b_{1i}$. And $\alpha_1, \ldots, \alpha_c \in \{h_1, \ldots, h_N\}$ is a regular sequence, then $(\alpha_1, \ldots, \alpha_c) : I \in P_c$.

Conjecture 1. 1. If $I \in P_c$, then the Herzog class of S/I (in particular $\beta_i(S/I)$) is determined by λ, N.

2. Every codim c licci ideal is in the same Herzog class as some $I \in P_c$.

3. If $I_1, I_2 \in P_c$ have different (λ, N), then they are in different Herzog classes.

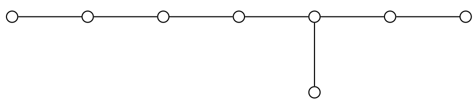
Conjectures 1, 2, and 3 are known for $c = 3$. Conjectures 1 and 2 are known if the following associated graph is ADE:



is ADE.

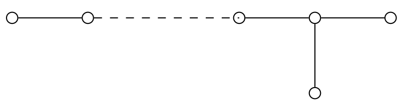
Examples based on (c, d, t) parameters:

$(c, d, t) = (6, 2, 1)$ yields the E_8 diagram:



E_8

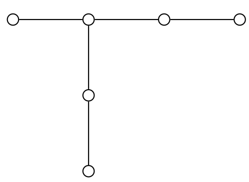
$(c, d, t) = (7, 2, 1)$ yields a diagram that is not Dynkin (E_{10}):



"E_{10}" Not Dynkin Diagram

$(c, d, t) = (4, d, 1)$ yields unspecified graphs.

$(c, d, t) = (4, 2, 2)$ yields a diagram that is not Dynkin ($E_6^{(1)}$):



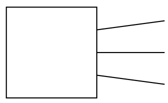
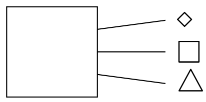
$E_6^{(1)}$ not Dynkin Diagram.

Corresponding to $\beta = (1, 6, -, -, 2)$.

Lecture 2 — Stability Patterns in free resolutions of symmetric ideals by Kartik Ganapathy

$$I_\infty = \langle x_i^2 x_j \mid 1 \leq i, j \leq 3 \rangle$$

Theorem 2 (Murai)**.** $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$
Each I_n is monomial & stable under the action of $S_n \curvearrowright k[x_1, \ldots, x_n]$. $\exists N$ s.t.

$\beta(I_N) =$  *and* $\beta(I_{N+1}) =$ 

Betti-table

Problem 1. *Interpret Murai's result algebraically. $\rightarrow$ Given an ideal I, what contributes the lines of a fixed slope?*

Theorem 3 (Cohen '67)**.** *Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a chain of homogeneous symmetric ideals, with $I_n \subseteq k[x_1, \ldots, x_n]$. Then $\exists f_1, f_2, \ldots, f_r \in I_N$ s.t.*

$$I_{N+t} = \langle S_{N+t} \cdot f_i \mid 1 \leq i \leq r, \forall t \geq 0 \rangle$$

Given such a chain, define

$$H_I(s,t) = \sum_{n \geq 1} H_{I_n}(t) \cdot s^n$$

By Nagel-Römer: This is a rational function in s & t.

Nagpal-Snowden have studied S_∞-stable ideals in $k[x_1, x_2, \ldots, x_n, \ldots]$, & also S_∞-equivariant modules on $k[x_1, \ldots, x_n, \ldots]$.

Conjecture 2 (Le-Nagel-Nguyen-Römer)**.** (i) $\exists a \in \mathbb{N}$ s.t. $\operatorname{pdim}(I_n) = n - a$ for $n \gg 0$

(ii) $\exists b, c \in \mathbb{N}$ s.t. $\operatorname{reg}(I_n) = bn + c$ for $n \gg 0$

By Auslander-Buchsbaum formula, $\operatorname{pdim}(I_n) + \operatorname{depth}(I_n) = n$

$$\implies \operatorname{depth}(I_n) = a \text{ for } n \gg 0$$

Idea: For fixed $i \geq 0$, endow $\{\operatorname{Ext}^i_{R_n}(k, I_n)\}$ with some algebraic structure with an $\mathbb{N}$-grading, & prove for finitely generated results

$$\text{endow } \{H^i_{m_n}(I_n)\} \curvearrowright$$

Definition 2. k an infinite field, of char $p > 0$. $V = k^\infty = \langle e_1, e_2, \ldots, e_n, \ldots \rangle$.
$GL = GL_\infty = GL(k^\infty)$; $R = \operatorname{Sym}(V) = k[x_1, x_2, \ldots]$
A GL_∞-equivariant R-module is an R-module M with an action of GL_∞ such that:

(i) $g(rm) = g(r) \cdot g(m) \quad \forall r \in R, m \in M, g \in GL \quad$ ($R \otimes M \rightarrow M$ is GL-equiv)

(ii) M is a smooth representation of GL_∞ (in char p subquotients of Schur functions)
M is a polynomial/smooth GL_∞-representation, and $M_n = M^{GL_{\infty - n}}$ is an R_n-module.
$M \longleftrightarrow M_n = M^{GL_{\infty - n}}$ will be a module over R_n.

Definition 3. Fix $e \in \mathbb{N}$, $\Gamma_e : \mathrm{Mod}_R \to \mathrm{Mod}_R$

$$M \longmapsto \{x \in M \mid (m^{[p^e]})^n \cdot x = 0 \text{ for } n \gg 0\}$$

Taking 0^{th} local-cohomology with respect to $m^{[p^e]}$.
The definition is meaningful since $m^{[p^e]}$ & $m^{[p^d]}$ are not copotent if $d \neq e$.

$\Gamma : \mathrm{Mod}_R \to \mathrm{Mod}_R$

$$M \longmapsto \{x \in M \mid \exists \text{ GL-stable ideal } 0 \neq I \subseteq R \text{ s.t. } I^n \cdot x = 0 \text{ for } n \gg 0\}$$

Theorem 4. *Let M be a f.g. GL-equiv R-module. Then $\forall e \in \mathbb{N}$, the right derived functor $R\Gamma_e(M)$ is also a f.g. GL-equiv R-module. & $(R\Gamma(M))$*

Theorem 5. *(i) The non-linear lines occurring in $\beta(M)$ all have slope of the form $p^e - 1$ with e allowed to vary.*

(ii) $R\Gamma_e(M)$ has pure slope $p^e - 1$ resolution agrees with that of M eventually.

Definition 4. We say a $\mathbb{Z}$-graded k-alg R is Koszul if the resolution of R/m looks like:

$$0 \leftarrow k \leftarrow R \leftarrow R(-1)^{b_1} \leftarrow R(-2)^{b_2} \leftarrow \cdots$$

Weighted rational curves:

Example 3. w.r.c. of type $(3,2)$
$D = [0:1]$
$\mathbb{P}^1 \hookrightarrow \mathbb{P}(1,1,1,2,2)$

$$[s,t] \mapsto [\underbrace{s^3 : s^2t : st^2}_{\substack{\text{deg } d \\ \text{monomials} \\ \text{vanishing at} \\ D}} : \underbrace{st^5 : t^6}_{\substack{\text{rest of} \\ \text{deg } d.e. \\ \text{monomials}}}]$$

$$S = k[x_0, x_1, x_2, y_0, y_1] \xrightarrow{\phi} k[s,t]$$

$$x_0 \mapsto s^3 \quad x_1 \mapsto s^2t \quad x_2 \mapsto st^2 \quad y_0 \mapsto st^5 \quad y_1 \mapsto t^6$$

$$I = \ker \phi \qquad R = S/I$$

Facts:

- $\odot$ I is determinantal,

$$I = I_2 \begin{pmatrix} x_0 & x_1 & x_2^2 & y_0 \\ x_1 & x_2 & y_0 & y_1 \end{pmatrix}$$

- $\odot$ R is Cohen-Macaulay.

What about Koszul property?

$$0 \leftarrow k \leftarrow R \leftarrow \begin{matrix} R(-1)^3 \\ \oplus \\ R(-2)^2 \end{matrix} \leftarrow \cdots$$

Definition 5 (Herzog-Reiner-Welker)**.** A $\mathbb{Z}$-graded k-alg R is nonstandard Koszul if $\operatorname{gr}_m R$ is Koszul.

Recall:

$$\operatorname{gr}_m R = \underbrace{R/m}_{0} \oplus \underbrace{m/m^2}_{1} \oplus \underbrace{m^2/m^3}_{2} \oplus \cdots$$

Example 4. $R = k[x,y]/(y^2 - x^3)$. $\operatorname{gr}_m R = k[x,y]/(y^2)$ is Koszul. So, R is non-standard Koszul.

Example 5. $R = k[x,y]/(y^2 - x^3 - x^6)$. $\operatorname{gr}_m(R) = k[x,y]/(y^2)$ is Koszul. So R is non-standard Koszul. (R - coordinate ring of a weighted rational curve).

Question 1. Is R nonstandard Koszul?

$$\operatorname{gr}_m(R) = k[x_0, x_1, x_2, y_0, y_1] \Big/ I_2 \begin{pmatrix} x_0 & x_1 & 0 & y_0 \\ x_1 & x_2 & y_0 & y_1 \end{pmatrix}$$

In fact, minors are a (quadratic) Gröbner basis (GB). So, R is nonstandard Koszul.

Theorem 6 (Davis, Sabionka '24)**.** *If R is the coordinate ring of a weighted rational curve. Then,*

(1) I is determinantal.

(2) R is Cohen-Macaulay.

(3) R is non-standard Koszul.

Lecture 3 — The Lefschetz properties for some modules over polynomial rings by Bek Chase

Definitions and Basics

Let k be a field of characteristic 0. Let $S = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring.

Let $A = S/I = \bigoplus_{i=0}^{c} A_i$ be an Artinian algebra, and let $M = \bigoplus_{i=a}^{b} M_i$ be an Artinian module over S.

Definition 6. The **Hilbert Function** of M is given by $i \mapsto \dim_k M_i$.
The **Hilbert Series** of M is given by $\operatorname{HS}(M) = \sum_{i=a}^{b} (\dim_k M_i) t^i$.

Definition 7. M has the **weak Lefschetz Property (WLP)** if there exists $\ell \in S_1$ such that multiplication by ℓ,

$$\times \ell : M_i \to M_{i+1}$$

has maximal rank for all i.

Definition 8. M has the **strong Lefschetz Property (SLP)** if multiplication by ℓ^d,

$$\times \ell^d : M_i \to M_{i+d}$$

has maximal rank for all i and for all d.

Fact. *1. If M has monomial generators, then $\ell = x_1 + \cdots + x_n$ is enough to check.*

2. If M has WLP/SLP, then $\mathrm{HS}(M)$ is unimodal.

Theorem 7 (Stanley, 1980)**.** *The quotient $k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$ has the SLP.*

Remark 2. This is a monomial complete intersection $\implies$ regular sequence, $\dim(S/I) = 0$.

Main Questions

1. Do all complete intersections have the SLP?
    - Note that $\mathrm{soc}(A) = (0 : \mathfrak{m})$, dim = 1-type. If $n = 4$, no!
2. Do all Gorenstein algebras have the SLP?
    - If $n = 4$, no!
3. What other algebras/modules have the SLP?

Consider Type 2 Artinian Algebras:

$$A_F = S/\mathrm{ann}(F) = \{P \mid P \circ F = 0\}$$

where F is a Macaulay dual generator and $\circ$ is differentiation.

- If F is a monomial $\implies$ A_F is a monomial complete intersection $\implies$ SLP.
- If F is a binomial $\implies$ $A_F = ?$

Proposition 1 (ADFMMRSV)**.** *Let $n = 3$. Then for all F binomial, $\mathrm{in}_<(A_F)$ is one of the following:*

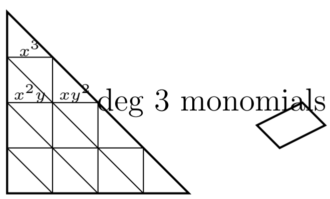
1. is a monomial complete intersection (type 2).

2. $(x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$.

3. an intersection of (1) and (2).

Biannular Regions and Central-Simple Modules

Let $I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma)$. WLP is characterized by Cook II, Nagel [07] $\implies$ biannular regions, lozenge tilings.



Central-Simple Modules [HW, 07]

Let A be an Artinian k-algebra, and $\ell \in A_1$. Let $P = \min\{\text{integer } s \text{ s.t. } \ell^P = 0\}$. We have the filtration:

$$A = (0 : \ell^P) + (\ell) \supseteq (0 : \ell^{P-1}) + (\ell) \supseteq \cdots \supseteq (0 : \ell^0) + (\ell) = (\ell)$$

Definition 9. The i-th Central-Simple Module (CSM) is:

$$V_{i,\ell} = \frac{(0 : \ell^i) + (\ell)}{(0 : \ell^{i-1}) + (\ell)}$$

Let $\tilde{V}_\ell = \bigoplus_{i=1}^{P} V_{i,\ell}$.

Fact. $\mathrm{HS}(\tilde{V}_\ell) = \mathrm{HS}(\mathrm{Gr}_{(\ell)}A) = \mathrm{HS}(A)$ *for* $0 \le i \le P$.

Theorem 8 (HW '07, Chase '24)**.** *A has SLP* $\iff \exists \ell \in A_1$ *such that* $\tilde{V}_\ell$ *has the SLP.*

Now apply this to the specific ideal: Let $I = (x^a, y^b, z^c, x^\alpha z^\gamma, y^\beta z^\gamma)$ and $\ell = z$. Then the Central-Simple Modules of A with respect to ℓ are:

$$V_{1,z} = \frac{(0 : z^1) + (z)}{(0 : z^0) + (z)} \cong \frac{k[x,y]}{(x^a, y^b)}$$

$$V_{2,z} \cong \frac{(x^\alpha, y^\beta)}{(x^a, y^b)}(-\alpha - \beta) \otimes \frac{k[z]}{(z^{\gamma_i})}$$

Remark 3. Tensor products do not always have the SLP. Example: $(1,3,3,1) \otimes (1,3,1,1)$.

Theorem 9 (Chase '24)**.** *Let* $I \subset k[x,y]$ *be a homogeneous Artinian ideal. Let* $J = (x^a, y^b) \subseteq I$. *Then* I/J *has the SLP.*

Remark 4. (S/I) has SLP $\iff$ $(S/\mathrm{in}_<(I))$ has SLP.

Lindström-Gessel-Viennot and Lattice Paths

The Lindström-Gessel-Viennot lemma relates determinants to lattice paths:

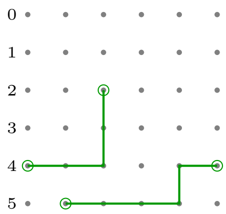
$$\det \begin{pmatrix} \text{matrix of} \\ \text{bin coefficients} \end{pmatrix} = \text{enumeration of lattice paths}$$

Recall the expansion:

$$\ell^d = (x+y)^d = \binom{d}{0} x^d + \binom{d}{1} x^{d-1} y + \dots$$

Example matrix evaluation:

$$\det \begin{pmatrix} \binom{3}{0} & \binom{3}{1} \\ \binom{3}{2} & \binom{3}{3} \end{pmatrix} \longrightarrow \begin{pmatrix} \binom{5}{2} & \binom{4}{2} & \binom{3}{2} \\ \binom{5}{4} & \binom{4}{4} & \binom{3}{4} \\ & 0 & \end{pmatrix} = 0$$



Theorem 10 (Chase '24). *S/I has the SLP if:*

1. $\alpha + \beta \leq a + b - c \leq \alpha + \beta + 1$
2. $\min(\alpha, \beta) \neq \max(\alpha, \beta) = \min(\alpha + \beta, a, b) \implies$ *S/I has a symmetric HS.*
3. $\sim\sim\sim$

where $I = \mathrm{in}_<(A_F)$.

Theorem 11 (ADFMMRSV). *For any A_F with $n = 3$, if F is binomial, then A_F has the SLP.*

Lecture 4 — Modules of Finite Length & Finite Projective Dimension by Nawaj KC

Q: How large can the annihilator of an R-module of finite pdim be?
Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq 0$ an R-module.

$$\mathrm{ann}_R(M) = \{r \in R \mid r \cdot M = 0\} \subseteq \mathfrak{m}$$

Theorem 12 (Auslander-Buchsbaum-Serre). *If $M \neq 0$ is an R-module, with $\mathrm{ann}_R(M) = \mathfrak{m}$ and yet $\mathrm{pdim}_R M < \infty$, then R is regular.*

Remark 5. $\mathfrak{m}M \subsetneq M \iff M \otimes_R k \neq 0$.

Theorem 13 (New Intersection Theorem). *("large annihilator") If $M \neq 0$ is an R-module with $\mathrm{ann}_R M \supseteq \mathfrak{m}^n$ for some $n \geq 1$ and yet, $\mathrm{pdim}_R M < \infty$, then R is Cohen-Macaulay.*

Goal: Quantify these theorems.

$$U_R(M) = \min\{i \mid \mathfrak{m}^i \subseteq \mathrm{ann}_R(M)\}$$

Small $U_R(M) \iff$ large annihilator.
Question: $\min\{U_R(M) \mid M \neq 0, \mathrm{pdim}_R M < \infty\} = ?$

Guiding Philosophy: Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $\underline{x}$ a maximal regular sequence.
Note that $R/(\underline{x}) = R/(x_1, \ldots, x_d)$ has:

1. Finite length: $\mathfrak{m}^n \subseteq (x_1, \ldots, x_d) \implies \mathrm{ann}_R(R/(x_1, \ldots, x_d))$ is large.
2. $\mathrm{pdim}_R(R/\underline{x}) = d < \infty$.

Such quotients by system of parameters (s.o.p.'s) are "simplest" or "smallest" R-modules of finite length & finite projective dimension.

Koszul complex: $\operatorname{Kos}(x_1, \dots, x_d) \xrightarrow{\simeq} R/(x_1, \dots, x_d)$

Conjectures: Let $M \neq 0$ be an R-module with $\ell_R(M) < \infty$ ($\implies U_R(M) < \infty$) and $\operatorname{pdim}_R M < \infty$. Then,

1. $\beta_i(M) \geq \beta_i(R/(\underline{x}))$ for some, hence any s.o.p. $\underline{x}$. [B-E-M]
2. $\ell_R(M) \geq \min\{\ell_R(R/(\underline{x})) \mid \underline{x} \text{ s.o.p.}\}$. [I-M-N]
    - Open in the local case.
    - For cyclic modules in $\dim R \leq 2$ [Nawaj - Andrew].
3. $U_R(M) \geq \min\{U_R(R/(\underline{x})) \mid \underline{x} \text{ s.o.p.}\}$. [C-H-P-R]

Remark 6. Conjecture (3) is true when R is Gorenstein & $\operatorname{gr}_{\mathfrak{m}}(R)$ is C.M. [A-B-I-M - 2010]. Assume $R/\mathfrak{m}$ is infinite.

Theorem 14 (Nawaj, Pollitz '25)**.** *Conj (3) holds, when* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay.*

Remark 7. $R = k[[x_1, \dots, x_n]]_{\mathfrak{m}}/(f_1, \dots, f_t) \rightsquigarrow \operatorname{gr}_{\mathfrak{m}}(R)$ is C.M. $\implies$ R is C.M.

$\implies \underline{x} = x_1, \dots, x_d \in \mathfrak{m} \setminus \mathfrak{m}^2$ sufficiently general.
$\implies \underline{x}^* = x_1^*, \dots, x_d^*$ in $\operatorname{gr}_{\mathfrak{m}}(R)$ is a regular sequence.

Key Lemma

$$1 \longmapsto S$$

$$\begin{array}{ccc} k & \longrightarrow & R/(x_1, \dots, x_d) \\ \uparrow \simeq & & \uparrow \\ F & \xrightarrow{\sigma} & K \end{array}$$

where S = socle element of maximal order.
$S \in \mathfrak{m}^n$ but $\mathfrak{m}^{n+1} = 0$ in $R/(x_1, \dots, x_d)$.
$n = \operatorname{ord}(S) \implies U_R(R/(x_1, \dots, x_d)) = n + 1$.

We have $\sigma(F) \subseteq \mathfrak{m}^n K$.

M is an R-mod.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_{d+1} & \longrightarrow & F_d & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \sigma_d & & & & \downarrow \sigma_0 & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & K_d & \longrightarrow & \cdots & \longrightarrow & K_0 & \longrightarrow & 0 \end{array}$$

$\sigma \otimes_R M : F \otimes_R M \to K \otimes_R M$.

Key notes: If $U_R(M) \leq n$ (i.e., $\mathfrak{m}^n \subseteq \operatorname{ann}_R(M)$):

$$\begin{array}{ccccccccc}
F_{d+1} \otimes_R M & \longrightarrow & F_d \otimes_R M & \longrightarrow & \cdots & \longrightarrow & F_0 \otimes_R M & \longrightarrow & 0 \\
\downarrow & & \downarrow {\scriptstyle \sigma_d \otimes M} & & & & \downarrow {\scriptstyle \sigma_0 \otimes M} & & \\
0 & \longrightarrow & K_d \otimes_R M & \longrightarrow & \cdots & \longrightarrow & K_0 \otimes_R M & \longrightarrow & 0
\end{array}$$

Then $\sigma \otimes_R M = 0$ map.

$$\operatorname{Cone}(\sigma \otimes_R M) = \operatorname{cone}(F \otimes_R M \xrightarrow{0} K \otimes_R M) = K \otimes_R M \oplus \Sigma F \otimes_R M$$

Also notice that $\operatorname{cone}(\sigma) \simeq_R R/(\underline{x}, S)$.

From the equalities:

$$\operatorname{Cone}(\sigma \otimes_R M) = \operatorname{cone}(\sigma) \otimes_R M$$

Taking homology gives:

$$H_{d+1}(\operatorname{cone}(\sigma) \otimes_R M) = H_{d+1}(K \otimes_R M) \oplus H_d(F \otimes_R M)$$

We know $H_{d+1}(K \otimes_R M) = 0$, so:

$$\operatorname{Tor}_{d+1}(R/(\underline{x}, S), M) = \text{Since } M \text{ has finite length, } \operatorname{depth}_R(M) = 0.$$

$$\text{If } \operatorname{pd}_R(M) < \infty \text{ and } R \text{ is CM of dim } d, \text{ then Auslander–Buchsbaum gives } \operatorname{pd}_R(M) = d.$$
$$\text{Hence } \operatorname{Tor}_d^R(k, M) \neq 0.$$

$\implies \operatorname{pd}_R M = \infty$.

Proof Sketch: Assume $\operatorname{pd}_R M < \infty$, $U_R(M) < \infty$.
Then, $U_R(M) \geq n + 1 = \min\{U_R(R/(\underline{y})) \mid \underline{y} \text{ s.o.p.}\}$.

Some new idea! (?)

$$\begin{array}{ccc}
k & \longrightarrow & R/(x_1, \ldots, x_d) \\
\uparrow & & \uparrow \\
F & \xrightarrow{\quad \sigma \quad} & K
\end{array}$$

$\sigma_d : F_d \to K_d = R$.