Rayfest: a conference in celebration of the mathematics of Raymond C. Heitmann

University of Nebraska-Lincoln (2025)

Lecture 1 — Finding the Nilradical by Craig Huenke

Let $S = K[x_1, \dots, x_n]$ or $K[[x_1, \dots, x_n]]$. Let $I \subseteq S$ be an ideal.
Goal: Find $\sqrt{I}$? (Eisenbud, Huneke, Vasconcelos)

We know:

$$\sqrt{I} = \bigcap_{\substack{P \supseteq I \\ P \text{ prime}}} P$$

$$\sqrt{I} = \{f \in S \mid f^N \in I \text{ for some } N\}$$

(*) means that one can characterise $\sqrt{I}$ as follows:

$$\sqrt{I} = \{f \in S \mid \forall Q \colon S \to K \text{ field s.t. } Q(I) = 0,\ Q(f) = 0 \text{ as well}\}$$

(Note: $Q \subseteq S/P$, where $P \supseteq I$).

Example 1. Let $S = K[X_{ij}]_{1 \le i,j \le n}$ and let $I = I_1(X^n)$. What is $\sqrt{I}$? → characteristic polynomial of X. If $\bar{X}$ is nilpotent, its characteristic polynomial is just T^n. So, $c_1 = \bar{X}_{11} + \cdots + \bar{X}_{nn}$, …, $c_n = \det(X)$ are all 0. So, $\sqrt{I} = \sqrt{(c_1, \dots, c_n)}$. Is $\sqrt{I} = (c_1, \dots, c_n)$?

How can one tell if $f \in \sqrt{I}$?

Theorem 2 (Kollár)**.** *Let $I = (f_1, \dots, f_k) \subseteq S$ of degree $d_1 \ge d_2 \ge \cdots \ge d_k > 2$. Then,*

1. *If $k \ge n$, then $(\sqrt{I})^{d_1 \cdots d_{n-1} d_k} \subseteq I$.*
2. *If $k < n$, then $(\sqrt{I})^{d_1 \cdots d_k} \subseteq I$.*

Practice:

$$f \in \sqrt{I} \iff 1 \in (IS[t], 1 - ft).$$

(This is the Rabinowitsch trick.) Also,

$$S_f \cong \frac{S[t]}{(1 - ft)},$$

so S_f is an S-algebra essentially of finite type.

Rothschild-Baumend: Let $R = \mathbb{C}[[f_1, \dots, f_n]] \subseteq \mathbb{C}[[x_1, \dots, x_n]] = S$. Let $P \in \operatorname{Spec}(R)$. Assume $S/\sqrt{PS}$ is regular, is R/P regular? ($x_i^d R = P, d \ge 2$)

Example 3 (Baby Example)**.** $f \in S = K[x]$ with $\operatorname{char}(K) = 0$. Find $\sqrt{(f)}$?

S is a U.F.D $\Rightarrow f = f_1^{a_1} \ldots f_d^{a_d}$ with $(f_i, f_j) = 1$.

$$\sqrt{(f)} = (f_1 \ldots f_d)$$

Consider $f' = f_1^{a_1-1} \ldots f_d^{a_d-1} g$ with $(g, f) = 1$.

$$(f) : f' = (f : f_1^{a_1-1} \ldots f_d^{a_d-1}) = (f_1 \ldots f_d) = \sqrt{(f)}$$

Example 4 (General-Up Example)**.** $I = (f_1, \ldots, f_n) \subseteq K[x_1, \ldots, x_n]$ with $\operatorname{char}(K) = 0$. Suppose $\sqrt{I} = (x_1, \ldots, x_n)$. How can we compute $\sqrt{(f_1, \ldots, f_n)}$?

Ideally, we want to find $\operatorname{soc}(S/I)$; if g generates the socle, then:

$$(I : g) = (x_1, \ldots, x_n) = \sqrt{I}$$

Scheja-Storch: $\det \left(\frac{\partial f_i}{\partial x_j} \right)$ is such a g.

Warning: This fails if $(f_1, \ldots, f_n)$ are not a regular sequence. Take $I = (x^2, xy, y^2)$.

$$\text{Socle} \left(\frac{\partial f_i}{\partial x_j} \right) = \begin{pmatrix} 2x & y & 0 \\ 0 & x & 2y \end{pmatrix}$$

Taking the 2×2 minors (assuming $\operatorname{char}(K) = 0$):

$$I_2 \left(\frac{\partial f_i}{\partial x_j} \right) = (x, y)^2 = I$$

Theorem 5 (Eisenbud-Huneke-Vasconcelos)**.** *Let $I \subseteq K[x_1, \ldots, x_n] = S$ with $I = (f_1, \ldots, f_k)$; let $d = \dim S/I$. Assume I is unmixed (all minimal primes have the same height) and I is generically a complete intersection (I_P has $\operatorname{ht} P$ generators $\forall P \in \operatorname{Min}(S/I)$). Then,*

$$\sqrt{I} = I : I_{n-d} \left(\frac{\partial f_i}{\partial x_j} \right)$$

In general, there are three remaining problems:

1. Can one reduce to I equidimensional?
2. Can one reduce to I having no embedded primes?
3. Can one reduce to I generically a complete intersection?

(1) & (2) are not much of a problem.

Example 6. If $I \subseteq S$ of $\operatorname{codim} c$ and $J = \bigcap$ primary components of I of $\operatorname{codim} c$. Then, $J = \operatorname{ann}(\operatorname{Ext}_S^c(S/I, S))$.

(*Hint:* show if $\underline{x}$ is a maximal regular sequence in I, then $J = \underline{x} : (\underline{x} : I) = \operatorname{ann}(\operatorname{Ext}^c(S/I, S))$).

Rather than finding a J such that $\sqrt{I} = (I : J)$, it suffices to find a J such that:

$$I \subsetneq (I : J) \subseteq \sqrt{I}$$

Test Example: $f_{mn} = x_1^n + \cdots + x_m^n$; $I_{mn} = (f_{m2}, f_{m3}, \ldots, f_{mn})$. Find $\sqrt{I_{mn}}$.

Theorem 7 (Vasconcelos)**.** *Let $S^b \xrightarrow{\phi} S^n \to I \to 0$. Then $I_1(\phi) \subseteq I \iff I$ is generated by a regular sequence. (Note: I_1 refers to the 1^{st} minors).*

Proof Sketch. $\Leftarrow$: (Koszul resolution)
$\Rightarrow$: $\otimes S/I$ yields:

$$(S/I)^b \xrightarrow{0} (S/I)^n \to I/I^2 \to 0$$

This implies I/I^2 is a free S/I module. By Vasconcelos, this implies I is generated by a regular sequence. □

Notice, $\mu(I_P) \leq k \iff I_{n-k}(\phi) \notin P$.
If I is unmixed of codim $= c$, then I is generated by a complete intersection $\iff \operatorname{ht}(I_{n-c}(\phi)) \geq c+1$.

Theorem 8. *Let $S = K[x_1, \ldots, x_n]$ and $I \subseteq S$. Let*

$$S^b \xrightarrow{\phi} S^n \to I \to 0$$

be a presentation of I, and let $c = \operatorname{ht}(I)$. Assume I is <u>not</u> generated by a complete intersection. Choose k maximal such that

$$J := I_{n-k}(\phi)$$

has height c. Then

$$I \subsetneq (I : J) \subseteq \sqrt{I}.$$

Algorithm: (Repeat).

Lecture 2 —Extremal Ideals & Properties of Square-free Monomial Ideals by Susan Morey

Background & Definition

Let $R = K[x_1, \ldots, x_n]$ where $a \in \mathbb{N}$, and $I = (m_1, \ldots, m_a)$, where the m_i are square-free monomials. Denote $[n] = \{1, 2, \ldots, n\}$.

Definition 9. Given $a \in \mathbb{N}$, define the polynomial ring

$$S_{[a]} = K[Y_A \mid A \subseteq [a];\ A \neq \emptyset]$$

Remark 10. This means we introduce a new variable Y_A for every non-empty subset A of $\{1, 2, \ldots, a\}$. For a set of size a, there will be $2^a - 1$ such variables.

Example 11. For $a = 3$,

$$S_{[3]} = K[Y_{\{1\}}, Y_{\{2\}}, Y_{\{3\}}, Y_{\{1,2\}}, Y_{\{1,3\}}, Y_{\{2,3\}}, Y_{\{1,2,3\}}]$$

We define elements $\varepsilon_i \in S_{[a]}$ by:

$$\varepsilon_i = \prod_{\substack{i \in A \\ \emptyset \neq A \subseteq [a]}} Y_A$$

Example 12. For $i = 2$ in the case of $a = 3$:

$$\varepsilon_2 = Y_{\{2\}}Y_{\{1,2\}}Y_{\{2,3\}}Y_{\{1,2,3\}}$$

To simplify notation, we will write Y_{12} for $Y_{\{1,2\}}$, so $\varepsilon_2 = Y_2Y_{12}Y_{23}Y_{123}$.

Let $\mathcal{E}_a = (\varepsilon_1, \dots, \varepsilon_a)$ be an ideal in $S_{[a]}$. Then $\mathcal{E}_a^r$ is generated by $\varepsilon_1^{r_1} \cdots \varepsilon_a^{r_a}$ where $\sum r_i = r$. We are often interested in the quotient ring $S_{[a]}/\mathcal{E}_a$.

Definition 13. For each $k \in [n]$, define $A_k = \{j \in [a] \mid x_k \text{ divides } m_j\}$.

Example 14. Let $I = (x_1x_2x_5,\ x_2x_5x_6,\ x_1x_3x_4x_6)$. Here $a = 3$ and $n = 6$. Let $m_1 = x_1x_2x_5$, $m_2 = x_2x_5x_6$, and $m_3 = x_1x_3x_4x_6$. Then:

$$\begin{aligned} A_1 &= \{1,3\} \\ A_2 &= \{1,2\} = A_5 \\ A_3 &= \{3\} = A_4 \\ A_6 &= \{2,3\} \end{aligned}$$

Definition 15. For each ideal I, define a ring homomorphism $\Psi_I : S_{[a]} \to R$ by:

$$\Psi_I(Y_A) = \begin{cases} \prod_{A_k = A} x_k & \text{if some such } k \text{ exists} \\ 1 & \text{otherwise} \end{cases}$$

Remark 16. The homomorphism Ψ_I maps a variable Y_A to the product of all original variables x_k that divide exactly the subset of monomials indexed by A.

Example 17. Continuing with the previous ideal I:

$$\begin{aligned} \Psi_I(Y_{\{1,3\}}) &= x_1 \\ \Psi_I(Y_{\{1,2\}}) &= x_2x_5 \\ \Psi_I(Y_{\{3\}}) &= x_3x_4 \\ \Psi_I(Y_{\{2,3\}}) &= x_6 \end{aligned}$$

For sets A that do not match any A_k, the map yields 1:

$$\Psi_I(Y_{\{1\}}) = \Psi_I(Y_{\{2\}}) = \Psi_I(Y_{\{1,2,3\}}) = 1$$

Facts.

1. *$\Psi_I(Y_A)$ is a square-free monomial (or 1) with disjoint support.*
2. *$\Psi_I(\varepsilon_i) = m_i \implies \Psi_I(\mathcal{E}_a) = I$.*

Example 18. Using the ε_2 computed earlier:

$$\begin{aligned} \Psi_I(\varepsilon_2) &= \Psi_I(Y_2Y_{12}Y_{23}Y_{123}) \\ &= \Psi_I(Y_{\{2\}})\Psi_I(Y_{\{1,2\}})\Psi_I(Y_{\{2,3\}})\Psi_I(Y_{\{1,2,3\}}) \\ &= 1 \cdot (x_2x_5) \cdot x_6 \cdot 1 \\ &= x_2x_5x_6 \\ &= m_2 \end{aligned}$$

Facts.

1. *$\beta_i(I^r) \leq \beta_i(\mathcal{E}_a^r)$.*
2. *$\Psi_I(\mathcal{E}_a) = I$.*
3. *$\Psi_I(\mathcal{E}_a^r) = I^r$.*

Primes and Intersections

Let P be a monomial prime in $S_{[a]}$. So $P = (Y_{B_1}, \dots, Y_{B_t})$ for some subsets $B_i \subseteq [a]$ where $B_i \neq \emptyset$.

Applying Ψ_I to P:

$$\Psi_I(P) = \left(\prod_{B_1 = A_k} x_k, \dots, \prod_{B_t = A_l} x_l \right) = \bigcap_{k : A_k = B_i} (x_{k_1}, \dots, x_{k_l}) = \bigcap P_i$$

where P_i is a monomial prime of R with $\operatorname{ht}(P_i) = \operatorname{ht}(P)$. (If the intersection is over an empty set, $\Psi_I(P) = R$).

If $P = (Y_{B_1}, \dots, Y_{B_t})$, then $\Psi_I(P^r) = \bigcap (x_{k_1}, \dots, x_{k_l})^r$.

Fact. *For monomials $M, N \in S_{[a]}$,*

$$\Psi_I(\operatorname{lcm}(M, N)) = \operatorname{lcm}(\Psi_I(M), \Psi_I(N))$$

For monomial ideals J, K of $S_{[a]}$,

$$\Psi_I(J \cap K) = \Psi_I(J) \cap \Psi_I(K)$$

The map Ψ_I takes elements of $\operatorname{Ass}(S_{[a]}/\mathcal{E}_a^r)$ to $\begin{cases} R \\ \operatorname{Ass}(R/I^r) \end{cases}$

Applications

Applications. *1.* ***Persistence:*** *If $P \in \operatorname{Ass}(\mathcal{E}_a^r) \implies P \in \operatorname{Ass}(\mathcal{E}_a^{r+1})$; then if it doesn't blow up (i.e., $\Psi_I(P) \neq R$), then $\Psi_I(P)$ is also in both.*

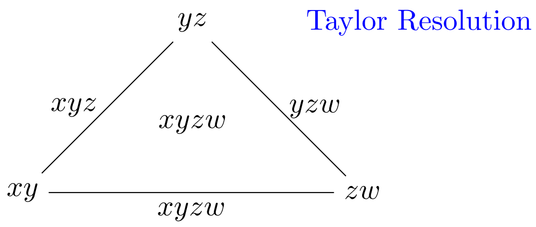
2. ***Index of Stability (***astab***):*** $\operatorname{astab}(I) \leq \operatorname{astab}(\mathcal{E}_a)$.

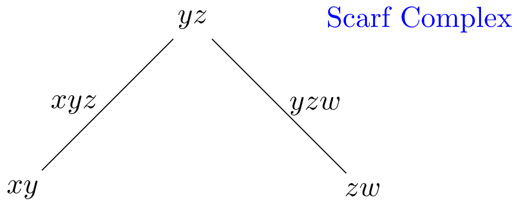
Take a partition of $[a] \longleftrightarrow$ Minimal prime of $\mathcal{E}_a$.

Example 19. For $a = 3$:

$$\{\{1\}, \{2, 3\}\} \longleftrightarrow (Y_1, Y_{23})$$

Example 20. Consider the ideal $I = (xy, yz, zw)$. We can construct the Taylor resolution and the Scarf complex for this ideal.





Remark 21. The Taylor resolution supports a homological complex mapping between free modules.

$$0 \to R \to \underset{\uparrow\text{edges}}{R^3} \to \underset{\uparrow\text{vertices}}{R^3} \to \cdots$$

In this example, $\mathcal{E}_a, \mathcal{E}_a^2, \mathcal{E}_a^3$ are Scarf complexes which support a minimal free resolution.

Betti Numbers Comparison

Below is a table comparing the Betti numbers for Taylor resolutions, the best known bounds before, and Scarf complexes for a specific case.

$r = 3$, $a = 6$			
	Taylor	**Best known before**	**Scarf**
β_3	367,290	230,360	19,845
β_4	3,819,816	2,118,790	58,530
β_{20}	1.3×10^{15}	6.7×10^{13}	0

Lecture 3 — Tor, Complete Intersections & Minors by David Jorgensen

Remark 22. In the following notes, "f.g." stands for finitely generated, and "RLR" stands for Regular Local Ring.

Theorem 23 (Auslander (1960))**.** *If M, N are f.g. modules over an unramified Regular Local Ring, then*

$$\operatorname{Tor}_i^R(M,N) = 0 \implies \operatorname{Tor}_j^R(M,N) = 0 \quad \forall j \geq i.$$

Theorem 24 (Lichtenbaum (1966))**.** *Rigidity of Tor holds for f.g. modules over any RLR.*

Conjecture (Peskine - Szpiro (1972))**.** If M, N are f.g. R-modules, with $\operatorname{pd}_R M < \infty$; then,

$$\operatorname{Tor}_i^R(M,N) = 0 \iff \operatorname{Tor}_j^R(M,N) = 0 \quad \forall j \geq i.$$

Theorem 25 (Heitmann (1993))**.** *There exists an affine k-algebra R & f.g. R-modules M, N with* $\operatorname{pd}_R M = 2$ *&* $\operatorname{length}(N) = 3$*, with* $\operatorname{Tor}_1^R(M,N) = 0$*, but* $\operatorname{Tor}_2^R(M,N) \neq 0$.

Specifically, let

$$0 \to R^2 \xrightarrow{X} R^4 \xrightarrow{Y} R^8$$

be a generic resolution. $\ell(M \otimes N) = 1$.

$$\operatorname{Im}(X \otimes N) = mN^4$$
$$\operatorname{Im}(Y \otimes N) = mN^8$$

This yields the following complex annotated with lengths:

$$\begin{tikzcd}
0 \arrow[r] & N^2 \arrow[r, "X\otimes N"] & N^4 \arrow[r, "Y\otimes N"] & N^8 \arrow[r] & \text{lengths} \\
& 6 \arrow[u] \arrow[ur, "\text{Not exact}"'] & 12 \arrow[u] \arrow[ur, "\text{exact (by counting lengths)}"'] & 24 \arrow[u] &
\end{tikzcd}$$

Theorem 26 (Jorgensen (1995))**.** *Let R be a complete intersection, M, N f.g. R-modules with* $r = \operatorname{cx}_R(M,N)$*. Then,* $\operatorname{Tor}_i^R(M,N) = 0$ *for $r+1$ consecutive values of $i > \dim R$. Then,* $\operatorname{Tor}_i^R(M,N) = 0$ $\forall i > \dim R$.

Next logical Step:

Theorem 27 (Bergh, Jorgenson, Thompson ('24))**.** *Let $(\mathcal{T}, \Sigma)$ be a triangulated category admitting a central ring action from a graded ring, $R = R^0[R^d]$ and X, Y objects of $\mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}^{\geq 0}(X, Y)$ is f.g. over R. Assume either,*

1. *$\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ has finite length over R^0 for all $n \gg 0$.*
2. *R^0 contains an infinite field or is local with infinite residue field.*

Then, $\exists m_0 \in \mathbb{Z}$ s.t. if $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^{n_i} Y) = 0$ for a complete set of residues, $n_1, \ldots, n_d \geq m_0 \pmod{d}$, then, $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0\ \forall n \geq m_0$.

Remark 28 (Bergh, Jorgensen, Thompson)**.** Introduce multiplicity, $e(X, Y)$ in $\mathcal{T}$. If $e(X, Y) = 0$, can assume only half vanishing, $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$.

Tor persistence

$$\operatorname{Tor}_i^R(M, M) = 0\ \forall i \gg 0 \implies \operatorname{pd}_R M < \infty$$

Theorem 29 (Asani ('24))**.** *Let X be an $r \times c$ matrix of indeterminants with $r \leq c$. $R = K[X]/I_r(X)$. Then, $\operatorname{Tor}_2^R(M, M) \neq 0$ where $M = \operatorname{coker} X$.*

Complete Intersections

Definition 30. R is a complete intersection if $R = Q/I$, where Q is a regular local ring and I is generated by a regular sequence.

Definition 31 (Grothendieck)**.** R is a complete intersection if $\widehat{R} = Q/I$, where Q is a regular local ring and I is generated by a regular sequence.

Question (Avramov)**.** Are complete intersections complete intersections?

$\implies$ Sometimes!

Theorem 32 (Heitmann, Jorgensen)**.** *If R is a one-dimensional integral domain, then it is a CI, if it is a $\widehat{CI}$.*

Theorem 33 (Heitmann, Jorgensen)**.** *$\exists$ a 3-dim local ring R, such that $\widehat{R} = Q/I$. Q regular local ring, I generated by regular sequence, but R is not a homomorphic image of a regular local ring. (R is not excellent).*

Minors

Theorem 34 (Buchsbaum - Eisenbud (1974))**.** *Let*

$$0 \to F_n \xrightarrow{A_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{A_1} F_0$$

be a finite free resolution. Then, R-Noetherian.

$$I_r(A) = \begin{bmatrix} c \\ o \\ l \\ u \\ m \\ n \end{bmatrix} \cdot \begin{bmatrix} r & o & w \end{bmatrix}$$

Theorem 35 (Heitmann (2024))**.** *Let R be not necessarily Noetherian, and let X be an $r \times c$ matrix of rank d. Assume $M_{\mu\nu}$ is a regular $d \times d$ minor. Then*

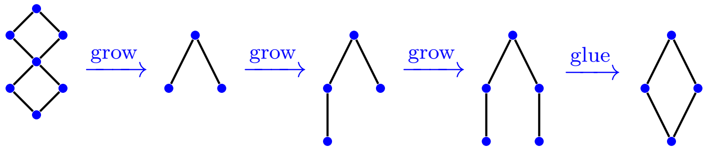
$$M_{\mu\nu} M_{\alpha\beta} = M_{\alpha\nu} M_{\mu\beta} \quad \forall \alpha, \beta.$$

Equivalently,

$$M_{\mu\nu} \begin{bmatrix} M_{\alpha_1\beta_1} & \cdots & M_{\alpha_1\beta_{\binom{c}{d}}} \\ \vdots & & \vdots \\ M_{\alpha_{\binom{r}{d}}\beta_1} & \cdots & M_{\alpha_{\binom{r}{d}}\beta_{\binom{c}{d}}} \end{bmatrix} = \begin{bmatrix} M_{\alpha_1\nu} \\ \vdots \\ M_{\alpha_{\binom{r}{d}}\nu} \end{bmatrix} \begin{bmatrix} M_{\mu\beta_1} & \cdots & M_{\mu\beta_{\binom{c}{d}}} \end{bmatrix}.$$

Lecture 4 — Heitmann type constructions by Susan Loepp

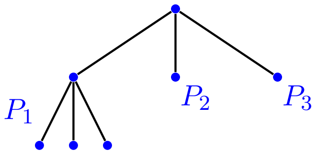
Theorem 36. *Let X be a finite poset. Then $\exists$ a quasi-excellent domain R s.t. X can be embedded into* Spec(R) *in a way that preserves saturated chains.*



Growing

Suppose X is a finite poset & R is a quasi-excellent local ring s.t. X can be embedded into Spec R in a way that preserves saturated chains. Let Y be a poset obtained from X by growing finitely many nodes out of a minimal node. Then, $\exists$ a quasi-excellent local ring S s.t. Y can be embedded into Spec(S) in a way that preserves saturated chains.

Notes on growing: R is quasi-excellent. $R[[Y, Z]]$. The poset X "lives" in $R[[Y, Z]]$ above $\langle Y, Z \rangle$.



$$J = \langle P_2, Y, Z \rangle \cap \langle P_3, Y, Z \rangle \cap \langle P_1, Y - \beta_1 Z \rangle \cap \langle P_1, Y - \beta_2 Z \rangle \cap \langle P_1, Y - \beta_3 Z \rangle$$

Gluing

Suppose X is a finite poset & R is a quasi-excellent ring s.t. X can be embedded into Spec(R) in a way that preserves saturated chains. Let Y be a poset obtained from X by gluing some minimal nodes together. Then, $\exists$ a quasi-excellent ring S s.t. you can embed Y into Spec S in a way that preserves saturated chains.

If $Z = \{P_1, \ldots, P_n\}$ be the set of minimal primes of R that you want to glue together. Then,

- $P_i \cap S = P_1 \cap S \quad \forall i = 1, \ldots, n$

- $\operatorname{Spec} R - Z \to \operatorname{Spec} S - \{P_1 \cap S\}$ given by $P \mapsto P \cap S$ is a bijection.

Theorem 37. *Let X be a finite poset. Then $\exists$ a Noetherian UFD A s.t. X can be embedded into* $\operatorname{Spec} A$ *in a way that preserves saturated chains.*

Notes. We know X can be embedded into $\operatorname{Spec} R$ for some Noetherian domain R.
Let $T = R[[x_1, \ldots, x_n]]$ when $n \geq \dim X$.
Then, X "lives" above $\langle x_1, \ldots, x_n \rangle$ since $T/\langle x_1, \ldots, x_n \rangle \cong R$. (where R is a subring of T)

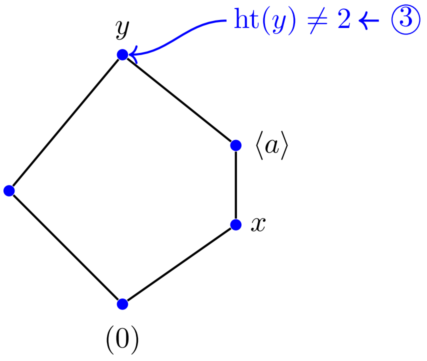
Now, use a Heitmann-type construction to construct A, a local UFD with $\widehat{A} = T$.

$$\dim A \geq 2$$

$\square$

Theorem 38. *Let X be a finite poset. Then $\exists$ a local UFD A s.t.* $\dim A = \dim X$ *& X can be embedded into* $\operatorname{Spec} A$ *in a way that preserves saturated chains* $\iff$

1. *X has exactly one maximum element.*
2. *X has exactly one minimum element.*
3. *If $x \in X$ is height 1 & $x < y$ is saturated, then* $\operatorname{ht}(y) = 2$.



Lecture 5 — Closure Operations in Characteristic 0 induced by resolutions of rings by Karl Schwede

History of Singularities

Char 0	**Char $p > 0$**
Log terminal	(Weakly) F-regular (all ideals are tightly closed)
Rational singularity	F-rational (all parameter ideals are tightly closed)

Theorem 39 (Smith, Hara, Mehta-Srinivas, Watanabe)**.** *Let* $R_{\mathbb{Z}}$ *be of finite type over* $\mathbb{Z}$*. Define*

$$R_{\mathbb{Q}} = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad R_p = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Then:

1. $R_{\mathbb{Q}}$ *has rational singularities if and only if* R_p *is* F*-rational for all sufficiently large* p*.*
2. *If* $R_{\mathbb{Q}}$ *is* $\mathbb{Q}$*-Gorenstein, then* $R_{\mathbb{Q}}$ *is log terminal if and only if* R_p *is strongly* F*-regular for all sufficiently large* p*.*

Definition 40 (Hochster-Huneke)**.** Let R be a domain of characteristic $p > 0$, and let $J \subseteq R$ be an ideal. We say $x \in J^*$ (the tight closure of J) if there exists $0 \neq c \in R$ such that under the Frobenius map ($R \to R^{1/p^e}$), we have:

$$c^{1/p^e} x \in J \cdot R^{1/p^e} \quad \forall e \gg 0.$$

Tight Closure Properties

(0) If R is regular, then $J^* = J$.

(1) Colon capturing: Let R be a local ring, and $x_1, \ldots, x_d$ be a system of parameters (s.o.p.). Then:

$$(x_1, \ldots, x_i) : x_{i+1} \subseteq (x_1, \ldots, x_i)^*$$

This implies that if R is a complete local ring, then $x \in J^* \iff \exists B$ a big Cohen-Macaulay R-algebra with $x \in JB$.

$$\frac{R}{(x_1,\ldots,x_i)} \xrightarrow{\cdot x_{i+1}} \frac{R}{(x_1,\ldots,x_i)}$$

Explanation: The kernel of the multiplication by x_{i+1} *map on the quotient module is exactly* $\frac{(x_1,\ldots,x_i):x_{i+1}}{(x_1,\ldots,x_i)}$*. Colon capturing controls this kernel by showing it lies within the tight closure.*

(2) Behavior under finite extensions $R \subseteq S$:

$$(JS)^* \cap R = J^*$$

(3) Briançon-Skoda Theorem: If $J = (f_1, \ldots, f_n)$ is an ideal generated by n elements, then the integral closure of powers of J satisfies:

$$\overline{J^{n+k-1}} \subseteq (J^k)^*$$

Definition 41 (Heitmann)**.** Let R be a mixed characteristic domain, and $J \subseteq R$. We define a closure operation where:

$$c^{\varepsilon} x \in \bigcap_n (J, p^n) R^+ \quad \forall \varepsilon > 0$$

Characteristic 0 Operations

Let $(R, \mathfrak{m})$ be a domain that is excellent with a dualizing complex. Let $\pi : Y \to \operatorname{Spec}(R)$ be a resolution of singularities (by Hironaka), so Y is non-singular.

Consider the derived global sections $R\Gamma(\mathcal{O}_Y)$. We have a derived pushforward $R\pi_*(\mathcal{O}_Y)$ which acts as a Cohen-Macaulay complex or dg-algebra. We have local cohomology properties:

$$\begin{aligned} &H^i_{\mathfrak{m}}(R\Gamma(\mathcal{O}_Y)) = 0 \quad \forall i < \dim(R) = d \\ &H^d_{\mathfrak{m}}(R\Gamma(\mathcal{O}_Y)) \neq 0 \end{aligned}$$

We can define closure operations based on these geometric constructions:

$$\ker\left(R \to H_0\left(\frac{R}{J} \otimes^L_R R\Gamma(\mathcal{O}_Y)\right)\right) = J^{\text{Hir}}$$

Question: Is $(J^{Hir})^{Hir} = J^{Hir}$*?*

For an ideal $J = (f_1, \ldots, f_c)$, using the Koszul complex $\operatorname{Kos}(f; R)$:

$$\ker\left(R \to H_0(\operatorname{Kos}(f; R) \otimes^L R\Gamma(\mathcal{O}_Y))\right) = J^{KH}$$

This closure operation satisfies $(J^{KH})^{KH} = J^{KH}$.

Properties of the KH Closure

(0) R has rational singularities $\iff$ all ideals are H_{ir} closed. This is also implied by ($\Leftarrow$) one full parameter ideal being KH closed.

(1) Colon capturing: For a system of parameters $(x_1, \ldots, x_d)$,

$$(x_1, \ldots, x_i)^{KH} : x_{i+1} \subseteq (x_1, \ldots, x_i)^{KH}$$

Also,

$$((x_1^a, x_2, \ldots, x_i)^{KH} : x_i^t) \subseteq (x_1^{a-t}, x_2, \ldots, x_i)^{KH}$$

(2) For a finite extension $R \subseteq S$: $(JS)^{KH} \cap R = J^{KH}$.

(3) For $J = (f_1, \ldots, f_n)$, we have $\overline{J^n} \subseteq J^{KH}$.

(4) It is computable in Macaulay2 (✓).

(5) The closure operation localizes.

General Closure Operations

A closure operation $I \mapsto I^*$ generally satisfies:

- $I \subseteq I^*$
- $I^{**} = I^*$
- $I \subseteq J \implies I^* \subseteq J^*$

Does it localize? Let $S = W^{-1}R$, is $I^*S = (IS)^*$? This is **false** for tight closure in general.

For an R-domain in characteristic $p > 0$:

$$x \in I^* \implies \exists c \neq 0 \text{ s.t. } cx^{p^e} \in I^{[p^e]} = \langle f_1^{p^e}, \ldots, f_s^{p^e} \rangle$$

A Way to Get a Closure Operation

Pick any module M. We can define a closure of an ideal I, denoted I^c, by taking the annihilator:

$$\begin{aligned} I^c &= \operatorname{ann}_R(M/IM) \\ &= \{x \mid xM \subseteq IM\} \\ &= \operatorname{ann}_R(R/I \otimes M) \\ &= \operatorname{ann}_R(\operatorname{coker}(I \hookrightarrow R) \otimes M) \end{aligned}$$

If $N \subseteq X$, we have a map $N \otimes M \to X \otimes M$.

If M is a maximal Cohen-Macaulay module (note that R itself may not be Cohen-Macaulay), we set $I^c = \operatorname{ann}_R(M/IM)$. Let $x_1, \ldots, x_d$ be a system of parameters on R (which may not be a regular sequence). Then we have colon capturing:

$$(x_1, \ldots, x_i) : x_{i+1} \subseteq (x_1, \ldots, x_i)^c$$

Note: Equality holds when $(x_1, \ldots, x_d)$ is a regular sequence.

Proof. Assume $u \cdot x_{i+1} \in (x_1, \ldots, x_i)$. Then:

$$\implies ux_{i+1}M \subseteq (x_1, \ldots, x_i)M$$

Since M is Cohen-Macaulay, the sequence $(x_1, \ldots, x_d)$ is regular on M. Thus, regular sequences smoothly pass through:

$$\implies uM \subseteq (x_1, \ldots, x_i)M \implies u \in (x_1, \ldots, x_i)^c$$

□

So, if we have such a closure operation with the colon-capturing property and $(x_1, \ldots, x_i)^c = (x_1, \ldots, x_i)$, this implies R is Cohen-Macaulay.

Tight closure has the colon-capturing property (using $M = R^+$). Furthermore, if R is a regular ring, then $I^* = I$.

Theorem 42 (Hochster–Roberts, in the usual Noetherian setting)**.** *If $R \to S$ is a pure extension of Noetherian rings and S is regular, then R is Cohen–Macaulay. In particular, a Noetherian direct summand of a regular ring is Cohen–Macaulay.*

Example 43. $R = k[x^3, x^2y, xy^2, y^3] \subseteq S = k[x, y]$.

Proof. It is enough to prove that $I^* = I$ in R. (This will intrinsically prove every system of parameters is regular).

We know $I^* \subseteq (IS)^* \cap R$. Assume $cx^q \in I^{[q]}$. Then:

$$\begin{aligned} cx^q \in (IS)^{[q]} \implies (IS)^* \cap R &= IS \cap R \quad \text{[since } S \text{ is regular]} \\ &= \operatorname{ann}_R(S/IS) \\ &= I \end{aligned}$$

By assumption, $S = R \oplus N$ as R-modules. Thus, the quotient splits:

$$S/IS \cong R/I \oplus N/IN$$

This implies $\operatorname{ann}_R(S/IS) = I$. Therefore, $I^* = I$. □

How to do Reduction to Characteristic p?

Let X be a variety over a field of characteristic 0. Suppose $I \subseteq \mathbb{C}[x_1, \ldots, x_n] = S$ and $X = \operatorname{Spec}(S/I)$. Since I has finitely many generators, only finitely many complex coefficients occur, say $\alpha_1, \ldots, \alpha_N$.

Choose a finitely generated $\mathbb{Z}$-subalgebra $A_0 \subseteq \mathbb{C}$ containing these coefficients, and set

$$R_{A_0} = \frac{A_0[x_1, \ldots, x_n]}{I_{A_0}},$$

where I_{A_0} is generated by the same polynomials. Then

$$R_{A_0} \otimes_{A_0} \mathbb{C} \cong S/I.$$

For a prime $\mathfrak{p} \in \operatorname{Spec}(A_0)$, the fiber

$$R_{\mathfrak{p}} := R_{A_0} \otimes_{A_0} \kappa(\mathfrak{p})$$

is a reduction of X to characteristic p. One then studies whether a given property holds for $R_{\mathfrak{p}}$ for all sufficiently general (often sufficiently large) primes p.

We can construct a finitely generated $\mathbb{Z}$-algebra:

$$A = \frac{\mathbb{Z}[\alpha_1, \ldots, \alpha_N][x_1, \ldots, x_n]}{I}$$

Let P_X be some statement about X. By reduction modulo p:

$$P(A \otimes_{\mathbb{Z}} \mathbb{Q}) = P(A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$$

This holds for a dense set of prime ideals $p \subseteq \operatorname{Spec}(\mathbb{Z})$.

Frobenius is Flat for Regular Rings

Consider the Frobenius map $F : R \to R$ given by $\Phi : x \mapsto x^p$. In characteristic p, the binomial expansion simplifies to $(x+y)^p = x^p + y^p$. For example, let $R = k[x, y]$. Then the image of the Frobenius map is the subring:

$$k[x^p, y^p] \subseteq k[x, y]$$