

# KUMUNU '25

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## Lecture 1 — Linkage and Licci Ideals By Craig Huneke

### Notation and Setup

Let  $(R, \mathfrak{m})$  be a regular local ring, or let  $R = K[x_1, \dots, x_n]$  be a polynomial ring over a field  $K$  with homogeneous maximal ideal  $\mathfrak{m} = (x_1, \dots, x_n)$ .

Let  $I \leq R$  be an ideal. In the homogeneous case, consider the minimal free resolution of  $R/I$ :

$$\cdots \longrightarrow \bigoplus_j R(-d_{cj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(-d_{1j}) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where  $c$  is the codimension of  $I$ . We denote the maximum and minimum degree shifts in the  $i$ -th step of the resolution as:

$$D_i = \max_j \{d_{ij}\} \quad \text{and} \quad d_i = \min_j \{d_{ij}\}.$$

Let  $I$  be a proper ideal. Consider a regular sequence  $\alpha_1, \alpha_2, \dots, \alpha_c$  inside  $I$ . We denote the ideal generated by this sequence as  $\underline{\alpha} = (\alpha_1, \dots, \alpha_c)$ .

**Definition 1** (Linkage). Let  $I, J \subsetneq R$  be proper ideals of the same codimension  $c$ . We say that  $I$  and  $J$  are linked by the regular sequence  $\underline{\alpha} = \alpha_1, \dots, \alpha_c \subseteq I \cap J$  if

$$J = (\underline{\alpha}) : I \quad \text{and} \quad I = (\underline{\alpha}) : J.$$

In this case we write  $I \sim_{\underline{\alpha}} J$ .

**Definition 2** (Peskine-Szpiro). We say two proper ideals  $I$  and  $J$  are linked via a regular sequence  $\underline{\alpha}$  if:

$$\underline{\alpha} : I = J \quad \text{and} \quad \underline{\alpha} : J = I.$$

**Definition 3.** Two ideals  $I_0$  and  $I_l$  are in the same **linkage class** if there exists a sequence of linkages:

$$I_0 \sim_{\alpha_0} I_1 \sim_{\alpha_1} \cdots \sim_{\alpha_{l-1}} I_l.$$

**Linkage via resolutions:** If  $I$  is linked to  $J$  via a complete intersection  $(\underline{\alpha}) = (\alpha_1, \dots, \alpha_c) \subseteq I$ , then  $J = (\underline{\alpha}) : I$ . A comparison map from a free resolution of  $R/I$  to the Koszul complex on  $\underline{\alpha}$  yields, via a mapping cone, a free resolution of  $R/J$ .

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_c & \longrightarrow & \cdots & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow a & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & \cdots & \longrightarrow & R^c & \longrightarrow & R/\underline{\alpha} & \longrightarrow & 0 \end{array}$$

$$J = \underline{\alpha} : I = I_1(a) + (\alpha_1, \dots, \alpha_c)$$

Alternatively,  $J = I^\vee$  (Alexander Dual).

*Remark 4.* In the special setting of squarefree monomial ideals, Alexander duality is a different construction that is often useful for studying resolutions and combinatorial properties. It should not be identified with linkage in general.

**Example 5.** Consider the ideal  $(x_1^2, \dots, x_n^2, J)$  where  $J$  is a square-free monomial ideal. Taking Frobenius powers  $x^{[q]}$  relates ideals of this form.

*Remark 6* (Hartshorne–Rao). For a locally Cohen–Macaulay curve  $C \subseteq \mathbb{P}^3$ , the Hartshorne–Rao module

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t))$$

is, up to shift, an invariant of the even liaison class of  $C$ .

## Licci Ideals

**Definition 7** (Licci). An ideal  $I$  is **licci** if it is in the linkage class of a complete intersection (i.e., a regular sequence).

Several necessary conditions must hold for an ideal to be licci:

**Theorem 8** (Buchweitz). *If an ideal  $I$  is licci, then  $I$  is strongly non-obstructed, i.e.,  $J/I^2 \otimes \omega_{R/I}$  is a Cohen-Macaulay module over  $R/I$ .*

**Theorem 9** (Huneke). *If an ideal  $I$  is licci, then  $I$  is strongly Cohen-Macaulay. In particular, the Koszul homology groups  $H_i(I; R)$  are Cohen-Macaulay over  $R/I$ .*

**Sufficient Conditions for Licci:** What about sufficient conditions for an ideal to be licci?

- 1) **Codimension 2:** If  $\text{codim } I = 2$ , then  $I$  is licci  $\iff R/I$  is Cohen-Macaulay.
- 2) **Codimension 3:** If  $\text{codim } I = 3$  and  $R/I$  is Gorenstein, then  $I$  is licci (J. Watanabe).  
Alternatively, let  $c = \text{codim } I$ , and define  $\delta = \mu(I) - c$ , where  $\mu(I)$  is the minimal number of generators. Let  $r = \text{type}(R/I)$  (the number of generators in the canonical module). If  $\delta = 1$ , then  $I$  is licci.

- 3) **Guerrieri-Ni-Weyman:** If  $c = 3$  and  $Q \subseteq R$ , then if  $(r \leq 2$  and  $\delta \leq 4)$  or  $(r \leq 4$  and  $\delta \leq 4)$ , then  $I$  is licci.

**Conjecture 10.** *If  $I$  is Cohen-Macaulay, then the following inequality implies  $I$  is licci:*

$$\frac{1}{c-1} + \frac{1}{r+1} + \frac{1}{\delta+1} > 1 \implies I \text{ is licci.}$$

**Conjecture 11** (Jelesiejew-Ramkumar-Samartano). *If  $I$  is of finite length, and  $[I] \in \text{Hilb}_\lambda^{SM}(\mathbb{A}^3)$ , then  $[I]$  is a smooth point if and only if  $I$  is licci. This is known to be true if  $I$  is a monomial ideal. (We can get information about an ideal being licci from its Betti table).*

## m-primary Monomial Ideals

Let  $I$  be an  $\mathfrak{m}$ -primary monomial ideal. We can write  $I$  in the form:

$$I = (x_1^{a_1}, \dots, x_n^{a_n}) + I^*$$

where  $I^*$  is the non-pure part.

**Theorem 12** (Huneke-Ulrich). (a) *If  $\text{codim } I^* \geq 2$ , then  $I$  is not licci.*

(b) *If  $\text{codim } I^* = 1$ , write  $I^* = x_1^{b_1} \cdots x_n^{b_n} K$ . Then  $I$  is doubly linked to the ideal  $(x_1^{a_1-b_1}, \dots, x_n^{a_n-b_n}) + K$ . This process can be repeated.*

**Example 13.** Let  $I = (a^4, b^4, c^4, ab^2c^2, b^2c^3, a^3b^2)$ . We factor out common powers and find that  $I$  is doubly linked ( $\sim\sim$ ) to:

$$K = (a^3, b^2, c^3, ac^2).$$

Since  $K$  is licci, it follows that  $I$  is licci.

**Example 14** (Adam Boucher). Let  $J = (a^4, b^4, c^4, b^2c^3, a^2c^3, a^2b^3)$ . This ideal is *not* licci.

*Remark 15.* The ideals  $I$  and  $J$  in the two examples above have the exact same Betti table, showing that the Betti table alone does not completely determine if an ideal is licci.

## Broken Complete Intersections

Consider an ideal of the form  $I = (x_1^{a_1}, \dots, x_n^{a_n}) + x_1 K$ . Taking the colon ideal with respect to  $x_1$ , we get:

$$(I : x_1) = (x_1^{a_1-1}, x_2^{a_2}, \dots, x_n^{a_n}) + K$$

This yields a short exact sequence:

$$0 \longrightarrow R/(I : x_1) \longrightarrow R/I \longrightarrow R/(I, x_1) \longrightarrow 0$$

Notice that the rightmost term is a complete intersection (C.I.):

$$R/(I, x_1) \cong R/(x_1, x_2^{a_2}, \dots, x_n^{a_n}).$$

**Definition 16.** A quotient  $R/I$  is a **broken complete intersection** if there exists a chain of ideals:

$$R \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_s = I$$

where the  $J_k$  are principal ideals modulo  $I$ , such that successive quotients  $J_{i-1}/J_i$  are complete intersections of codimension  $c$ . That is:

$$J_{i-1}/J_i \cong R/(\text{regular sequence}).$$

**Theorem 17.** *The following implications hold for broken complete intersections:*

- *Broken C.I.  $\implies$  licci.*
- *Broken C.I.  $\longleftarrow$   $\mathfrak{m}$ -monomial licci.*

## Lecture 2 — Multiplicities & Reductions by Dan Katz

### Bigraded Hilbert Function

The bigraded Hilbert function is given by:

$$H(r, s) = \sum_{i+j=d} c_{ij} \binom{r}{i} \binom{s}{j}$$

The coefficients  $c_{ij}$  are precursors to “mixed multiplicities”.

### Historical Context

#### (1943, 1945) Chevalley

1. Let  $R$  be a complete local domain,  $K \subseteq R$ , with  $[R : K] < \infty$ , and let  $\underline{x}$  be a system of parameters (s.o.p.). The degree of the field extension provides a foundation:

$$[R : K[\underline{x}]] = e(\underline{x})[R : K]$$

This formulation naturally leads to the definition of multiplicities.

2. Extending this to Generalized Local Rings (GLR), the multiplicity is calculated as:

$$e(\underline{x}) = \sum e \left( \frac{\hat{R}}{\mathfrak{p}} \right) \lambda(\hat{R}_{\mathfrak{p}})$$

#### Chevalley (historical precursor):

Early work related multiplicity to finite extensions determined by systems of parameters, helping motivate the later formal definition of multiplicity.

#### Samuel (1947, 1949, 1951):

Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ , and let  $I \subseteq R$  be an  $\mathfrak{m}$ -primary ideal. Then for  $n \gg 0$ ,

$$\lambda(R/I^{n+1}) = P_I(n),$$

where  $P_I(n)$  is a polynomial of degree  $d$ . Its leading term is

$$P_I(n) = \frac{e(I)}{d!} n^d + \text{lower degree terms},$$

where  $e(I)$  is the Hilbert–Samuel multiplicity of  $I$ . If  $\underline{x}$  is a system of parameters with  $(\underline{x}) \subseteq I$ , then one has

$$e(\underline{x}) \geq e(I).$$

Here,  $d$  is defined as the least number of elements generating an  $\mathfrak{m}$ -primary ideal.

**Serre (early 50’s):**

For a system of parameters  $\underline{x}$ , the multiplicity can be expressed as the Euler characteristic of the corresponding Koszul complex  $K(\underline{x})$ :

$$\underline{x} = \text{s.o.p.}, \quad e(\underline{x}) = \chi(K(\underline{x})) = \sum_{i=0}^d (-1)^i \lambda(H_i(K(\underline{x})))$$

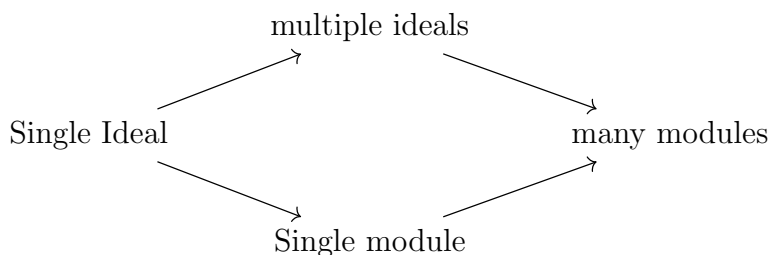
**Northcott-Rees (1954):**

- An ideal  $J \subseteq R$  is defined as a *reduction* of  $I$  if  $JI^n = I^{n+1}$  for some  $n$ .
- Reductions and minimal reductions are guaranteed to exist.
- The integral closures of the ideals match:  $\bar{J} = \bar{I}$ .

$$[x \in \bar{I} \iff x^n + i_1 x^{n-1} + \dots + i_n = 0, \quad i_j \in I^j]$$

**Generalizations**

The goal is to generalize these concepts from single ideals to more complex structures:



**1. Many Ideals**

**Bigraded Hilbert Polynomial**

For  $r, s \gg 0$ , the Hilbert function agrees with a polynomial

$$H(r, s) = \sum_{i+j \leq d} c_{ij} \binom{r}{i} \binom{s}{j}.$$

Its top-degree part is

$$\sum_{i+j=d} c_{ij} \binom{r}{i} \binom{s}{j},$$

and the coefficients in top degree lead to the mixed multiplicities.

**Risler–Teissier (1973):**

Let  $I_1, \dots, I_q$  be  $\mathfrak{m}$ -primary ideals in a local ring of dimension  $d$ . Then for  $n_1, \dots, n_q \gg 0$ ,

$$\lambda\left(\frac{R}{I_1^{n_1} \dots I_q^{n_q}}\right) = P(n_1, \dots, n_q),$$

where  $P$  is a polynomial of total degree  $d$ . Its top-degree part can be written as

$$\sum_{k_1 + \dots + k_q = d} \frac{e(I_1^{[k_1]}, \dots, I_q^{[k_q]})}{k_1! \dots k_q!} n_1^{k_1} \dots n_q^{k_q}.$$

The coefficients  $e(I_1^{[k_1]}, \dots, I_q^{[k_q]})$  are the mixed multiplicities.

**Most Important Case ( $q = d$ ):**

Set  $e(I_1 \dots I_d) = \text{coeff of } n_1 \dots n_d$ . This is “the” mixed multiplicity of  $I_1, \dots, I_d$  (the case for  $n = 2$  was studied by Bhattacharya).

**Rees (1984): “Joint Reductions”**

Let  $I_1, \dots, I_q$  be  $\mathfrak{m}$ -primary ideals, and pick elements  $x_i \in I_i$ . The sequence  $x_1 \dots x_q$  is a *joint reduction* if:

$$x_1 I_2 \dots I_q + I_1 \dots I_{q-1} x_q \text{ is a reduction of } I_1 \dots I_q.$$

If  $q = d$ , Rees proved:

$$e(I_1 \dots I_d) = e(\underline{x}) = \chi(K(\underline{x}))$$

## 2. Single Module

**Buchsbaum–Rim (1963):**

Let  $M \subseteq F$  be a submodule of a free  $R$ -module  $F$  of rank  $r$ , and assume  $\lambda(F/M) < \infty$ . For each  $n$ , the natural map

$$S_n(M) \longrightarrow S_n(F)$$

has image denoted  $R_n(M)$ . Then for  $n \gg 0$ ,

$$\lambda\left(\frac{S_n(F)}{R_n(M)}\right) = P(n),$$

where  $P(n)$  is a polynomial of degree  $d + r - 1$ . Its leading term is

$$\frac{br(M)}{(d + r - 1)!} n^{d+r-1},$$

where  $br(M)$  is the Buchsbaum–Rim multiplicity of  $M$ .

**Parameter module case:**

If  $\mu(M) = d + r - 1$ , then the Buchsbaum–Rim multiplicity can be expressed as the Euler characteristic of the Buchsbaum–Rim complex associated to a presentation of  $M$ .

**Rees (1987): Reduction of module.**

Write  $M^n$  for  $R_n(M)$ . A submodule  $N \subseteq M$  is a reduction if:

$$NM^n = M^{n+1}$$

Let  $\overline{M}$  be defined as the degree 1 component of  $\overline{R(M)}$  in  $S(F)$ .

### 3. Many Modules

Let  $M_i \subseteq F_i$  be submodules of free modules, with  $\lambda(F_i/M_i) < \infty$  for each  $i$ . Under suitable hypotheses, one studies multigraded length functions of the form

$$\lambda\left(\frac{F_1^{n_1} \cdots F_q^{n_q}}{M_1^{n_1} \cdots M_q^{n_q}}\right),$$

which agree for large  $(n_1, \dots, n_q)$  with a polynomial whose top-degree coefficients define mixed Buchsbaum–Rim multiplicities.

**(2015) Callejas-Bedregal + Perez:**

(i) Let  $M_1, \dots, M_q \subseteq F$  with  $\lambda(F/M_i) < \infty$ . We study the length function:

$$\lambda\left(\frac{F^{n_1+\cdots+n_q}}{M_1^{n_1} \cdots M_q^{n_q}}\right) = P(n_1, \dots, n_q), \quad n_i \gg 0$$

The degree is  $d + r - 1$ .

Let  $x_i \in M_i$ . The sequence  $x_1, \dots, x_q$  is a Joint Reduction (JR) if  $\sum x_i M_1 \cdots \widehat{M}_i \cdots M_q$  is a reduction of  $M_1 \cdots M_q$ .

(ii) Let  $M_1 \subseteq F_1, \dots, M_q \subseteq F_q$  with  $\lambda(F_i/M_i) < \infty$  and  $r_k = \text{rk}(M_k)$ .

**Definition 18** (Joint reduction). Let  $B_i \subseteq M_i$  for  $i = 1, \dots, q$ . One says that  $B_1, \dots, B_q$  form a joint reduction of  $M_1, \dots, M_q$  if

$$\sum_{i=1}^q B_i M_1 \cdots \widehat{M}_i \cdots M_q$$

is a reduction of  $M_1 \cdots M_q$ .

**Theorem 19** (DKV). 1. Joint reductions exist if  $\text{depth } R > 0$ .  $B_i$  can be taken to be free. If  $d = q$ , then  $\mu(B) \sim \mu(M_i)$ .

2. The following are equivalent (TFAE):

- (i)  $B_1, \dots, B_q$  is a JR.
- (ii) For all minimal primes  $P$  and discrete valuation rings (DVRs)  $R/P \subseteq V \subseteq \text{QF}(R/P)$ , we have  $B_x V = M_x V$ .
- (iii)  $\det B_1, \dots, \det B_q$  is a JR, comparing to  $I_1, \dots, I_q$  where  $I_j = I(M_j)$ .

## General Theory

$$\lambda \left( \frac{F_1^{n_1} \cdots F_q^{n_q}}{M_1^{n_1} \cdots M_q^{n_q}} \right) = P(n_1, \dots, n_q), \quad n_i \gg 0.$$

The total degree is given by  $d + r_1 + \cdots + r_q - q$ .

If  $q = d$ , then  $\text{deg} = r_1 + r_2 + \cdots + r_d$ . (Assuming  $B_i \subseteq M_i$ ).

**Definition 20.**  $br(M) = r_1! \cdots r_d! \cdot (\text{coefficient of } n_1^{r_1} \cdots n_d^{r_d})$ .

Let the following exact sequence define  $K(\phi_i)$ :

$$0 \rightarrow F_j \rightarrow F_i \rightarrow 0 \quad K(\phi_i)$$

**Theorem 21** (DKV).

$$br(M_1 \cdots M_q) = \chi(K(\phi)) = e(I_1 \cdots I_d)$$

where  $K(\phi) = \bigotimes_i K(\phi_i)$ .

### Case Study:

Let  $R$  be a 2-dimensional Regular Local Ring (RLR).

Let  $M_1, M_2$  be complete modules, and  $B_i \subseteq M_i$  be a Joint Reduction (JR).

Then we have the identity:

$$M_1 M_2 = M_1 B_2 + B_1 M_2$$

This leads to an **Explicit Description!** for the length function:

$$\lambda \left( \frac{F_1^{n_1} F_2^{n_2}}{M_1^{n_1} M_2^{n_2}} \right).$$