# Commutative Algebra in Mixed Characteristic

A conference in honor of Bhargav Bhatt's Rowlee Lecture-UNL

## 1 Lecture 1 — Rowlee Lecture: Bhargav Bhatt

**$p$-adic numbers:**

**1. $p \to p$-adic absolute value ($p$ fixed prime)**

$$\left|\frac{a}{b}\right|_p = \left|p^r \cdot \frac{a'}{b'}\right| = \frac{1}{p^r} \qquad \text{Ex: } p^n \to 0 \text{ as } n \to \infty$$

where $a', b'$ are coprime to $p$.

**Definition 1** (Hensel 1897)**.** $\mathbb{Q}_p$: completion of $(\mathbb{Q}, |\cdot|_p)$; can $+$, $-$, $\times$. We have $|\cdot|_p$ on $\mathbb{Q}_p$.

$$\mathbb{Q}_p = \left\{ \sum_{i=-N}^{\infty} a_i p^i \;\middle|\; a_i \in \{0, \ldots, p-1\} \right\} \quad (\text{“Formal Laurent series in } p\text{”})$$

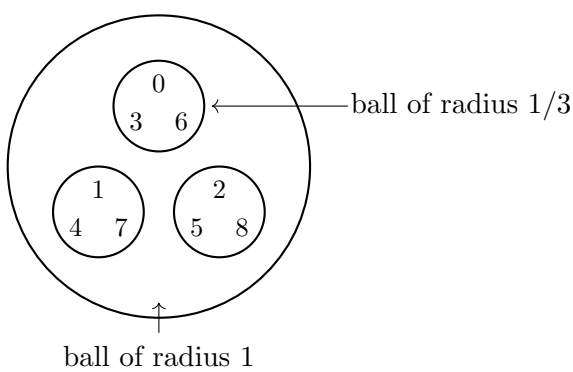
**Key Property:** For $a, b \in \mathbb{Q}_p$, $|a+b|_p \leq \max(|a|_p, |b|_p)$.

$$\mathbb{Z}_p = \{a \in \mathbb{Q}_p \mid |a|_p \leq 1\} \quad (\text{stable under } +, \times).$$

$\hookrightarrow$ PID, with unique prime $(p)$.

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{\text{kill } (p)} \mathbb{Z}_p \xrightarrow{\text{invert } p} \mathbb{Q}_p$$

Picture of $\mathbb{Z} \subset \mathbb{Z}_3$:



**Examples:**

**i) checking integrality:**

$$f(x) = \sqrt{1+x} = (1+x)^{1/2} = \sum_{i=0}^{\infty} \binom{1/2}{i} x^i \in \mathbb{Q}[[x]]$$

where,

$$\binom{t}{i} = \frac{t(t-1)\cdots(t-i+1)}{i!}$$

**Claim.** *All coeffs lie in* $\mathbb{Z}[1/2]$.

$\Rightarrow$ Fix $p$ odd; so, $1/2 \in \mathbb{Z}_p$. We have a poly map,

$$g : \mathbb{Q}_p \xrightarrow{t \mapsto \binom{t}{i}_{\mathbb{Q}_p}} \mathbb{Q}_p$$

$$g(\mathbb{Z}) \subset \mathbb{Z} \subset \mathbb{Z}_p.$$

By $p$-adic continuity,

$$g(\mathbb{Z}_p) \subset \mathbb{Z}_p \implies \binom{1/2}{i} \in \mathbb{Z}_p \implies p \nmid \text{denominator of } \binom{1/2}{i}.$$

**2) Finding Solutions:**

**Theorem 1** (Hasse–Minkowski, 1920s)**.** *Let $f(x_1, \ldots, x_n)$ be a quadratic form over $\mathbb{Q}$. Then $f$ has a nontrivial zero over $\mathbb{Q}$ if and only if $f$ has a nontrivial zero over $\mathbb{Q}_p$ for every prime $p$ and over $\mathbb{R}$.*

Compare (Selmer 65): Analog of $\Leftarrow$ fails for $3x^3 + 4y^3 + 5z^3$.

**3) Counting Solutions:**

$X =$ system of poly eqns in $n$-variables$/\mathbb{F}_p$.

**Theorem 2** (Weil, 1949)**.** *Let $X$ be a variety over $\mathbb{F}_p$. Its zeta function is*

$$Z(X,t) = \exp\left(\sum_{m \geq 1} \#X(\mathbb{F}_{p^m}) \frac{t^m}{m}\right).$$

*Then $Z(X,t)$ is a rational function.*

1) $Z(X,t)$ is a rational function. (Dwork, Grothendieck)

2) We know a great deal about the location of its zeros and poles. (Deligne)

# Recent Developments: $\mathbb{Q}_p \approx \mathbb{F}_p((t))$

**Fontaine-Wintenberger 1979:**

$$K = \mathbb{Q}_p(p^{1/p^\infty}) \qquad K^\flat = \mathbb{F}_p((t^{1/p^\infty}))$$

$$\mathrm{Gal}(\overline{K}/K) = \mathrm{Gal}(\overline{K^\flat}/K^\flat)$$

**Scholze 2012:** $\exists$ categories of "perfectoid spaces" over $K$ & $K^\flat$ which are naturally identified.

$$\begin{tikzcd}
\{\text{alg varieties}/K\} \arrow[d, dashed, "ART.FORM."'] & \{\text{alg varieties}/K^\flat\} \arrow[d, dashed, "\text{extract power roots}"] \\
\mathrm{Perf}_K \arrow[r, "\simeq"] & \mathrm{Perf}_{K^\flat}
\end{tikzcd}$$

**Theorem 3** (André 2016)**.** *Let $R = \mathbb{Z}[x_1, \dots, x_n]$. Suppose $R \xrightarrow{d} S$ is an injective map with $S$ a module finite / $R$. $\Rightarrow \exists$ $R$-linear map $\beta : S \to R$ s.t $\beta \circ d = id$.*

**Theorem 4** (C-A-K-N- 2024)**.**

**Theorem 5** (Antieau–Krause–Nikolaus, 2024)**.** *For fixed $n$, the groups $K_{2i}(\mathbb{Z}/p^n)$ vanish for all sufficiently large $i$.*

## 2 Lecture 2 — Connectedness of the associated graded ring- De Stefani

Let $(R, \mathfrak{m}, k)$ be a complete local ring. We can write $R = T/I$ where $T = V[[x_1, \dots, x_n]]$ and $V$ is an unramified DVR with uniformizer $p = \mathrm{char}(k)$. For example, $V = \mathbb{Z}_p$.

$$\begin{tikzcd}
 & k \\
V \arrow[ur, dash] \arrow[dr, dash] & \\
 & V_{(p)}
\end{tikzcd}$$

The associated graded ring is defined as:

$$G = \mathrm{gr}_{\mathfrak{m}_R}(R) = \mathrm{gr}_{\mathfrak{m}_T}(T)/\operatorname{in}(I) = \bigoplus_{i \geq 0} \mathfrak{m}_R^i / \mathfrak{m}_R^{i+1}$$

where $\mathrm{gr}_{\mathfrak{m}_T}(T) \cong k[X, \dots] = P$.

For $0 \neq f \in T$, if $f \in \mathfrak{m}^v \setminus \mathfrak{m}^{v+1}$, let $v = v(f)$ be the $\mathfrak{m}$-adic order of $f$. We define $\operatorname{in}(f)$ as the image of $f$ inside $\mathfrak{m}_T^v / \mathfrak{m}_T^{v+1}$. Then $\operatorname{in}(I) = (\operatorname{in}(f) \mid 0 \neq f \in I)$.

**Example 1.** $R = k[[t^6, t^7, t^{15}]] \cong \frac{k[[X,Y,Z]]}{(XZ - Y^3, Z^2 - X^5)}$
$G = \frac{k[X,Y,Z]}{(XZ, Z^2, Y^3Z, Y^6)}$

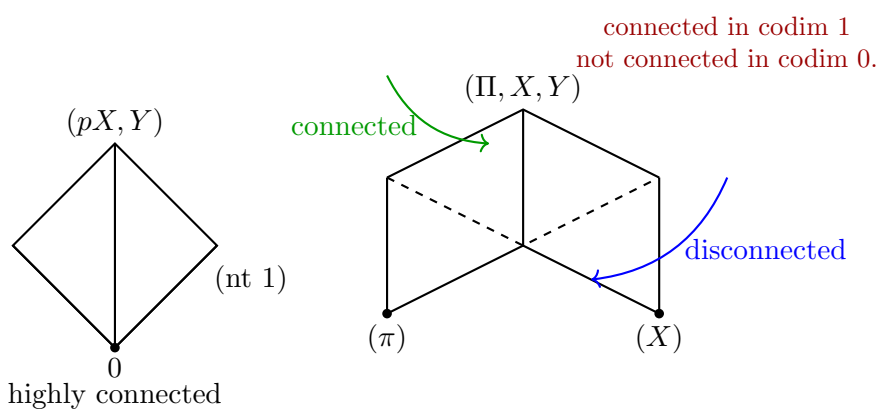
**Example 2.** $R = \frac{\mathbb{Z}_p[[X,Y]]}{(pX - Y^3)} \qquad G = \frac{\mathbb{F}_p[\Pi, X, Y]}{(\Pi X)}$

## Properties of localized graded rings

| Property of localized graded rings | $G \rightsquigarrow R$ | $R \rightsquigarrow G$ |
|---|---|---|
| Regular | ✓ | ✓ |
| Complete Intersection | ✓ | × (1) |
| Gorenstein | ✓ | × (1) |
| Cohen-Macaulay | ✓ | × (1) |
| Reduced | ✓ | × (1) |
| Domain | ✓ | × (2) |

**Example (2) revisited:** For $R = \frac{\mathbb{Z}_p[[X,Y]]}{(pX - Y^3)}$, we localize $G$:

$$G_{\mathfrak{m}} \cong \left( \frac{\mathbb{F}_p[\Pi, X, Y]}{(\Pi X)} \right)_{(\Pi, X, Y)}$$




**Definition 2** (Hartshorne '62)**.** Let $A$ be a ring with $d = \dim(A) < \infty$. $A$ is connected in codim $s$ $(0 \leq s \leq d)$ $\iff$ $\mathrm{Spec}(A) \setminus Z$ is connected $\forall Z \subseteq \mathrm{Spec}(A)$ closed with $\dim(Z) < d - s$. [$\emptyset$-disconnected, $\dim(\emptyset) = -1$]

**Easy remarks:**

- $A$ connected in codim $d$ $\iff$ $\mathrm{Spec}(A)$ connected.
- $(A, \mathfrak{m})$ local, connected in codim $d-1$ $\iff$ $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ connected.
- $A$ connected in codim 0 $\iff$ $A_{\mathrm{red}} = A/\sqrt{0}$ domain (remove minimal primes).

**Theorem 6** (Hartshorne '62)**.** $(A, \mathfrak{m})$ *satisfying Serre's* $(S_2)$ $\implies$ *$A$ connected in codim 1.*

**Remark 1.** $A$ is connected in codim 1 $\implies$ $A$ equidimensional.

**Theorem 7** (Hochster-Huneke '02)**.** $(A, \mathfrak{m})$ *complete equidimensional. $A$ connected in codim* 1 $\iff$ $H^d_{\mathfrak{m}}(A)$ *indecomposable.*

### "The affine case"

Let $P = k[x_1, \ldots, x_n]$, $J \subseteq P$, and let $<$ be a monomial order. Let $\operatorname{in}_<(J)$ be the monomial ideal in $P$.

**Conjecture 8** (Kredel-Weispfenning '88)**.** *$P/J$ domain $\implies$ $P/\operatorname{in}_<(J)$ equidimensional.*

**Theorem 9** (Kalkbrener-Sturmfels '95)**.** *Let $k = \overline{k}$. $P/J$ domain $\iff$ $P/\operatorname{in}_<(J)$ connected in codim 1. $\implies$ $P/J$ connected in codim 0.*

**Theorem 10** (Varbaro '09)**.** *Let $s > 0$. $P/J$ connected in codim $s$ $\implies$ $P/\operatorname{in}_<(J)$ is connected in codim $s$.*

**Theorem 11** (De Stefani, Ross, Varbaro)**.** *Let $(R, \mathfrak{m})$ be complete, $s > 0$. $R$ is connected in codim $s$ $\implies$ so is $G = \operatorname{gr}_{\mathfrak{m}}(R)$.*

**Ingredients for Varbaro's thm ['09]:**

$$\begin{tikzcd} k[t] \arrow[r] & \hom(P/J) \arrow[r, "t\neq 0"] \arrow[dr, "t=0"'] & P/J \\ & & P/\operatorname{in}_<(J) \end{tikzcd}$$

**Theorem 12** (Grothendieck)**.** *Let $(A, \mathfrak{m})$ be complete, $s > 0$, and let $t \in \mathfrak{m}$ be a parameter. $A$ connected in codim $s$ $\implies$ so is $A/(t)$.*

### Homogenization and Orders

Let $T = V[[x_1, \ldots, x_n]] \ni f \neq 0$. We have $\operatorname{in}(f) = (x)$-adic order of $f$. Over $(V, p)$, the maximal ideal is $\mathfrak{m}(T) = (\pi, x)$-order of $f$.

We construct:

$$T = \frac{S[[\pi]]}{(\pi - p)} \qquad S[[t]] = T[[\pi, t]] = B$$

For $F \in S$, $\hom(F) = t^{-\operatorname{in}(F)} \cdot F(x_1 t, \ldots, x_n t)$.

**Example 3.** Let $T = \mathbb{Z}_p[[X, Y]]$ and $I = (pX - Y^3) \ni f$. Let $S = \mathbb{Z}_p[[\pi, X, Y]]$ and $B = S[[t]]$. Then:

$$\hom(f) = t^{-3}(pXt - (Yt)^3) = pX - Y^3 t^2$$

Starting with $R$:

$$R = \frac{\mathbb{Z}_p[[X, Y]]}{(pX - Y^3)} \rightsquigarrow R = \frac{\mathbb{Z}_p[[\pi, X, Y]]}{(\pi - p, pX - Y^3)}$$

Let $I' = (\pi - p, pX - Y^3)$. We define $\hom(I') = (\hom(f) \mid 0 \neq f \in I')$.

$$\begin{aligned} \hom(\pi - p) &= \pi t - p \\ \hom(pX - Y^3) &= pX - Y^3 t^2 \\ \hom(\pi X - Y^3) &= \pi X - Y^3 t \end{aligned}$$

So the homogenization of $R$ is:

$$\hom(R) = \frac{\mathbb{Z}_p[[X, Y, \pi, t]]}{\hom(I')} = \frac{\mathbb{Z}_p[[X, Y, \pi, t]]}{(\pi t - p, pX - Y^3 t^2, \pi X - Y^3 t)}$$

$$\begin{tikzcd}
 & & R \\
\hom(R) = \text{Complete Local Ring} \arrow[urr, "\text{invert } t"] \arrow[drr, "t=0"'] & & \\
 & & \frac{\mathbb{Z}_p[[X,Y,\pi]]}{(p,\pi X)} = \widehat{G}
\end{tikzcd}$$

**Theorem 13.** *Let $(R, \mathfrak{m})$ be complete over $V$. Then:*

$$\hom(R)\widehat{\otimes}_{V[[t]]}V((t)) \cong R\widehat{\otimes}V((t))$$

*Moreover, if $R$ is connected in codim $s$, so is $\hom(R)$. The connectedness graph $\Gamma_s(R)$ is connected $\iff$ $\Gamma_s(\hom(R))$ is connected.*

# 3 Lecture 3 — Perfectoid Pure Singularities by Linquan Ma

Let $p$ be a fixed prime number.

**Definition 3.** Let $R$ be a Noetherian $\mathbb{F}_p$-algebra. The ring $R$ is defined to be:

- **F-pure** if the Frobenius map $R \xrightarrow{F} R$ is pure as a map of $R$-modules. (Recall that $R \to F_*R$ is pure $\iff$ $R \to F_*R$ splits, when $R$ is F-finite).
- **F-injective** if the induced map on local cohomology $H^i_{\mathfrak{m}}(R) \xrightarrow{F} H^i_{\mathfrak{m}}(R)$ is injective for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ and for all $i$.

**Remark 2.**
- $R$ is F-pure / F-injective $\iff$ so is $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ $\iff$ so is $R_Q$ for all $Q \in \mathrm{Spec}(R)$.
- We have the following implications:

$$\text{F-pure} \implies \text{F-injective} \implies \text{reduced + semi-normal}$$

  Furthermore, if $R$ is Gorenstein, F-injective implies F-pure.
- (Hara-Watanabe) F-pure and $\mathbb{Q}$-Gorenstein $\implies$ log canonical.

**Example 4.** (a) Let $R = K[x, y, z]/(f)$ such that $X = \mathrm{Proj}(R)$ is an elliptic curve. Then, $R$ is F-pure $\iff$ $X = \mathrm{Proj}(R)$ is ordinary $\iff$ $H^1(X, \mathcal{O}_X) \xrightarrow{F} H^1(X, \mathcal{O}_X)$ is injective.

(b) Let $R = K[x_1, \ldots, x_n]/(\text{sq-free monomial ideal})$. Then $R$ is F-pure. (Fedder's criterion: If $R = S/I$, where $S$ is a regular local graded ring, then $R$ is F-pure $\iff$ $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$).

(c) Let $R = K[x_1, \ldots, x_n]/I$. If $\mathrm{in}_<(I)$ is square-free, then $R$ is F-injective but not necessarily F-pure.

**Theorem 14** (Fedder, Horiuchi-Miller-Shimomoto)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local $\mathbb{F}_p$-algebra, and let $f \in \mathfrak{m}$ be a non-zero divisor. Suppose one of the following holds:*

1. *$R/(f)$ is Cohen-Macaulay and F-injective.*

2. *$R/(f)$ is F-pure.*

*Then, $R$ is F-injective. (In particular, if $R$ is Gorenstein and $R/(f)$ is F-pure, then so is $R$).*

$$\begin{tikzcd}
 & \tilde{R}/t \arrow[r, "\simeq"] & K[x]/\operatorname{in}_>(I) \\
K[t] \arrow[r, hook] & \tilde{R} \arrow[u] \arrow[d] & \\
 & \tilde{R} \otimes_{K[t]} K(t) \arrow[r, "\simeq"] & R \otimes_K K(t)
\end{tikzcd}$$

We note the equivalences:

$$R \text{ is F-pure} \iff R \to F_*R \text{ is pure} \iff R \to F_*^e R \text{ is pure } \forall e \iff R \to \varinjlim_e F_*^e R \text{ is pure.}$$

Here, $\varinjlim_e F_*^e R = R_{\text{perf}}$, which is a perfect ring.

**Definition 4** (Bhatt-Morrow-Scholze)**.** A ring $S$ is called **perfectoid** if:

1. $S$ is $p$-complete and $\exists \varpi \in S$ such that $\varpi^p = p \cdot u$ for some $u \in S^\times$.

2. $S/p \xrightarrow{\text{Frob}} S/p$ is surjective.

3. $W(S^\flat) \xrightarrow{\theta} S$, with $\ker(\theta)$ being a principal ideal.

Here, $S^\flat = \varprojlim_{\text{Frob}} S/p$.

**Example 5.** 1. Perfect rings of characteristic $p$. $W(S) \xrightarrow{\text{Pro}} S$, $\ker = (p)$.

2. $\mathbb{Z}_p[x_1, \ldots, x_d] \to \mathbb{Z}_p\left[p^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, \ldots, x_d^{\frac{1}{p^\infty}}\right]^{\wedge p}$ is a perfectoid ring.

   (If $S$ is $p$-torsion free, then $S$ is perfectoid $\iff$ $S/\varpi \xrightarrow{\text{Frob}} S/p$ is bijective).

(3) Let $R$ be a Noetherian complete local domain with $\operatorname{char}(R/\mathfrak{m}) = p > 0$. Then $(R^+)^{\wedge p} \longleftarrow$ perfectoid.

**Definition 5.** Let $R$ be a Noetherian ring such that $p \in J(R)$ (the Jacobson radical of $R$). Then, $R$ is **perfectoid pure** if $\exists$ a perfectoid $R$-algebra $B$ so that $R \to B$ is pure.

(We say $R$ is **perfectoid injective** if $\exists$ $B$ such that $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(B)$ is injective for all $\mathfrak{m}$, for all $i$).

**Example 6.** 1. In characteristic $p$, perfectoid pure $\iff$ F-pure, and perfectoid injective $\iff$ F-injective.

2. Regular local rings of mixed characteristic $p$ are perfectoid pure.

$$A = \mathbb{Z}_p[[x_1, \ldots, x_d]] \to A_\infty = A\left[p^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, \ldots, x_d^{\frac{1}{p^\infty}}\right]^{\wedge p} \quad \text{is faithfully flat.}$$

3. All splinters of mixed characteristic are perfectoid pure. $R \to R^+$ is pure ( $\implies R \to (R^+)^{\wedge p}$ is pure). E.g., summand of regular rings.

**Remark 3.**
- $R$ is perfectoid pure/injective $\iff$ so is $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ $\iff$ so is $R_Q$ for all $Q \in V(p)$.
- We have the implications:

$$\text{perfectoid pure} \implies \text{perfectoid injective} \implies \text{reduced + semi-normal.}$$

  Also, perfectoid injective + Gorenstein implies perfectoid pure.

**Theorem 15** (BMPSTWW)**.** *If $R$ is perfectoid pure and $\mathbb{Q}$-Gorenstein with index not divisible by $p$, then $R$ is log-canonical.*

**Theorem 16** (BMPSTWW)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $f \in \mathfrak{m}$ be a non-zero divisor. If $R/(f)$ is a complete intersection and perfectoid pure, then $R$ is perfectoid pure.*

**Example 7.** 1. Let $R = \mathbb{Z}_p[x, y, z]/(x^3 + y^3 + z^3)$. If $p \equiv 1 \pmod 3$, then $R$ is perfectoid pure. (Proof: $R/p \cong \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$ is F-pure $\square$).

1'. $R = \mathbb{Z}_p[x, y, z]/(x^3 + y^3 + z^3)$, $p \equiv 2 \pmod 3$. Yes, $R$ is still perfectoid pure.

**Theorem 17** (Yoshikawa)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local $p$-torsion free ring, which is a complete intersection. If $R/pR$ is quasi-F-split, then $R$ is perfectoid pure. (e.g., Cone of elliptic curve are always quasi-F-split).*

**Example 8.** 2. Let $R = \mathbb{Z}_p[x, y]/(p^3 + x^2 + y^3)$, with $p \equiv 1 \pmod 3$. $R$ is perfectoid pure.

$$\underbrace{\mathrm{gr}_{\mathfrak{m}}(R)}_{\text{F-pure}} \rightsquigarrow \underbrace{R[\mathfrak{m}t, t^{-1}]_{(\mathfrak{m})}}_{\text{perfectoid pure}} \rightsquigarrow \text{R is a summand of } R[\mathfrak{m}t, t^{-1}]_{(\mathfrak{m})} \implies \text{R is perfectoid pure.} \quad \square$$

2'. Let $R = \mathbb{Z}_p[x, y]/(p^3 + x^3 + y^3)$, with $p \equiv 2 \pmod 3$. $R$ is not perfectoid pure. (Brozzo, Tagathost-Pandey-Moreno-Schwede-Sridhar).

**Question:** Is there a Fedder type criterion for perfectoid pure singularities?

# 4 Lecture 4: Are Determinantal Rings Pure Subrings of Polynomial Rings? By Anurag Singh

**Definition 6.** A ring homomorphism $\phi\colon R \to S$ is **PURE** if $(R \to S) \otimes_R M$ is injective $\forall$ $R$-modules $M$.

- $R \hookrightarrow S$ faithfully flat $\implies$ pure.
- Let $V$ be a finite rank vector space over $\mathbb{C}$.
- Let $G$ be a finite subgroup of $\mathrm{GL}(V) \curvearrowright S = \mathrm{Sym}(V)$.

We have that $S^G \hookrightarrow S$ is faithfully flat $\iff$ $S^G$ is a polynomial ring. However, the inclusion $S^G \hookrightarrow S$ is always pure.

- If $R \hookrightarrow S$ is $R$-split, then it is pure.
- If $R$ is a complete local ring, or if $R, S$ are $\mathbb{N}$-graded finitely generated with $R_0 = S_0$ a complete local ring, then $R \hookrightarrow S$ is $R$-split $\iff$ pure.

Suppose $R$ is an $\mathbb{N}$-graded finitely generated ring over a field $R_0 = K$, and $R$ is a pure subring of a polynomial ring $K[X]$. Then:

(a) $R$ is normal.

(b) (Hochster-Roberts) $R$ is Cohen-Macaulay.

(c) (Boutot) In characteristic 0: $R$ has rational singularities.

(d) (Hochster-Huneke) $R$ is $F$-regular.

(e) (Smith) $R$ is simple as a $D_{R|K}$-module.

(f) (Smith-Van den Bergh) In characteristic $p$:

$$\begin{cases} R \text{ has FFRT} \\ D_{R|K} \text{ is a simple ring} \end{cases}$$

(g) (Huneke-Sharp, Lyubeznik) Fix an ideal $I$ in $R$. Then $\operatorname{Ass}_R H_I^k(R)$ is finite.

**Theorem 18** (Heitmann-Ma)**.** *For mixed characteristic $R \hookrightarrow S$, if it is pure with $S$ regular, then $R$ is pseudo-rational (in particular, Cohen-Macaulay).*

If $G$ is a linearly reductive subgroup of $\mathrm{GL}_n(K)$, acting on a polynomial ring $S = K[X]$, then $S^G \hookrightarrow S$ is pure.

## Determinantal Rings

Let $G = \mathrm{GL}_t(\mathbb{C}) \curvearrowright S = \mathbb{C}[Y_{m\times t}, Z_{t\times n}]$. The action is given by:

$$\begin{aligned} M\colon Y &\mapsto YM^{-1} \\ Z &\mapsto MZ \end{aligned} \implies YZ \mapsto YM^{-1}MZ = YZ \quad [m, n \geq t+1,\ t \neq 0]$$

This gives the isomorphism and inclusions:

$$\frac{\mathbb{C}[X_{m\times n}]}{I_{t+1}} \simeq \mathbb{C}[YZ] = \mathbb{C}[Y,Z]^{\mathrm{GL}_t} \subseteq \mathbb{C}[Y,Z]$$

## Symmetric Determinantal Rings

Let $G = O_t(\mathbb{C}) = \{M \in \mathrm{GL}_t : M^tM = \mathrm{id}\}$. The action is given by $M\colon Y \mapsto MY$, so $Y^tY \mapsto Y^tM^tMY = Y^tY$.

$$\begin{tikzcd} \mathbb{C}[Y]^{O_t} \arrow[r, hook] \arrow[d, equal] & \mathbb{C}[Y_{t\times n}] \\ \mathbb{C}[Y^tY] \arrow[d, "\simeq"'] & \\ \frac{\mathbb{C}[X^{\mathrm{Sym}}_{n\times n}]}{I_{t+1}} & \end{tikzcd} \qquad [n \geq t+1,\ t \neq 0]$$

## Pfaffian Determinantal Rings

Let $G = \mathrm{Sp}_{2t}(\mathbb{C}) = \{M \in \mathrm{GL}_{2t} : M^t \Omega M = \Omega\}$, where $\Omega = \begin{pmatrix} 0 & -\mathrm{id}_t \\ \mathrm{id}_t & 0 \end{pmatrix}$. The action is given by $M\colon Y \mapsto MY$. Thus, $Y^t \Omega Y \mapsto Y^t M^t \Omega M Y = Y^t \Omega Y$.

$$\frac{\mathbb{C}[X^{\mathrm{alt}}_{n\times n}]}{\mathrm{Pf}_{2t+2}} \simeq \mathbb{C}[Y^t \Omega Y] = \mathbb{C}[Y]^{\mathrm{Sp}_{2t}} \subseteq \mathbb{C}[Y] \quad [n \geq 2t+2,\ t \neq 0]$$

**Theorem 19** (Hochster-Jeffries-Pandey-Singh)**.** *Let $K$ be a field of characteristic $p > 0$.*

1. *$K[YZ] \subseteq K[Y,Z]$ is pure $\iff$ $t = 1$.*
2. *$K[Y^tY] \subseteq K[Y]$ is pure $\iff$ $\begin{cases} t = 1 & \text{or} \\ t = 2, & p \text{ odd} \end{cases}$*
3. *$K[Y^t\Omega Y] \subseteq K[Y]$ is pure $\iff$ Pigs Fly (?)*

**Question:** Is the LHS a pure subring of SOME polynomial ring?

**Theorem 20** (Jeffries, Singh)**.** *Assume $m, n \geq t+1$, and let $V$ denote $\mathbb{Z}$ or $\widehat{\mathbb{Z}_{(p)}}$. Then:*

1. *$V[YZ]$ is a pure subring of some polynomial ring over $V$ if and only if $t = 1$.*
2. *$V[Y^tY]$ is a pure subring of some polynomial ring over $V$ if and only if $t = 1$, or $t = 2$ and $V = \widehat{\mathbb{Z}_{(p)}}$ for $p$ odd.*
3. *$V[Y^t\Omega Y]$ is a pure subring of some polynomial ring over $V$ $\iff$ Pigs Fly.*

Let $Y$ be a $2 \times n$ matrix.

$$\frac{\mathbb{Z}[Y^tY]}{I_3} \simeq \frac{\mathbb{Z}[X^{\mathrm{Sym}}_{n\times n}]}{I_3} \simeq \mathbb{Z}[Y^tPY] \quad \text{where } P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Notice that $(Y^tPY)^t = Y^tP^tY = Y^tPY$. Let $Y = \begin{pmatrix} y_1 & \cdots & y_n \\ z_1 & \cdots & z_n \end{pmatrix}$. Then:

$$Y^tPY = \begin{pmatrix} y_1 & z_1 \\ \vdots & \vdots \\ y_n & z_n \end{pmatrix} \begin{pmatrix} z_1 & \cdots & z_n \\ y_1 & \cdots & y_n \end{pmatrix} = \begin{pmatrix} 2y_1z_1 & y_1z_2 + y_2z_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Now consider the rings over $\mathbb{Z}\left[\frac{1}{2}\right]$:

$$\mathbb{Z}\left[\frac{1}{2}\right][Y^tPY] = \mathbb{Z}\left[\frac{1}{2}\right][y_iz_i,\ y_iz_j + y_jz_i] \subseteq \mathbb{Z}\left[\frac{1}{2}\right][y_iz_j] \subseteq \mathbb{Z}\left[\frac{1}{2}\right][y_i, z_i]$$

Here, the group $S_0 = \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts via $\sigma\colon y_i \leftrightarrow z_i$. The subring $\mathbb{Z}\left[\frac{1}{2}\right][y_iz_j]$ corresponds to $S_0$-invariants, and $\mathbb{Z}\left[\frac{1}{2}\right][y_i, z_i]$ is the polynomial ring $S$. The inclusion $\mathbb{Z}\left[\frac{1}{2}\right][y_iz_j] \subseteq \mathbb{Z}\left[\frac{1}{2}\right][y_i, z_i]$ is pure (for degree reasons).

**Conclusion:** All inclusions are pure.