CA+ Conference

A two-day conference on commutative algebra and related fields at Iowa State

1 The Briançon-Skoda Theorem (Lecture by Linquan Ma)

1.1 Preliminaries and Background

Definition 1 (Recall: Integral Closure)**.** Let $I \subseteq R$ be an ideal. The integral closure of I, denoted by $\overline{I}$, consists of elements $x \in R$ so that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

for some n, and some $a_i \in I^i$.

Remark 2 (Geometric Interpretation)**.** Under mild assumptions, let $Y = \widetilde{Bl_I R}$ (the normalization of the blowup). Let $\pi \colon \widetilde{Bl_I R} \to \operatorname{Spec}(R)$ be the proper birational morphism. Then,

$$\overline{I^n} = \pi_* \mathcal{O}_Y(-nE)$$

where $\mathcal{O}_Y(-E) = I \cdot \mathcal{O}_Y$.

Definition 3 (Pseudo-rationality)**.** A local ring (R, m) is called **pseudo-rational** if R is Cohen-Macaulay, normal, $\widehat{R}$ is reduced, and for every proper birational map $Y \xrightarrow{\pi} \operatorname{Spec}(R)$, the natural map on local cohomology

$$h^d R\Gamma_m R \to h^d R\Gamma_m R\pi_* \mathcal{O}_Y$$

is injective. Equivalently, $\pi_* \omega_Y \cong \omega_R$.

Remark 4.

- R is regular $\implies$ R is pseudo-rational.
- If R is essentially of finite type over a field of characteristic 0, then pseudo-rational $\iff$ rational singularity.
- If R is excellent and $\operatorname{char}(R) = p > 0$, then F-rational $\implies$ R is pseudo-rational (the converse is not true).

Example 5. Let $R = \frac{k[[x,y,z]]}{(x^a + y^b + z^c)}$. Then R is pseudo-rational if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

1.2 Classical and Known Results

Theorem 6 (Briançon-Skoda, Lipman-Sathaye)**.** *Let R be regular, and let $I \subseteq R$ be an ideal that can be generated by n elements. Then,*

$$\overline{I^{n+k-1}} \subseteq I^k \quad \forall k \geq 1.$$

In particular (for $k = 1$), $\overline{I^n} \subseteq I$.

Remark 7. Suppose $J \subseteq I$ is a reduction (i.e., $\overline{J} = \overline{I}$), say $\mu(J) = l$. Then,

$$\overline{I^{l+k-1}} = \overline{J^{l+k-1}} \subseteq J^k \subseteq I^k.$$

In particular, using a minimal reduction, $\overline{I^{\dim(R)+k-1}} \subseteq I^k$ for all $I \subseteq R$ and for all $k \geq 1$.

Remark 8.

1. The original Briançon-Skoda theorem was for $R = \mathbb{C}\{x_1, \dots, x_d\}$.
2. Lipman-Sathaye extended it to the general case (R regular).
3. There exist other proofs via multiplier ideals, test ideals, tight closure, etc.

Theorem 9 (Huneke '90)**.** *Suppose R is reduced, essentially of finite type over a field or a mixed characteristic DVR. Then, there exists N depending only on R, so that*

$$\overline{I^{N+k-1}} \subseteq I^k \quad \forall I \subseteq R,\ \forall k \geq 1.$$

(Note: B.S. says we can take $N = \dim R$ if R is regular).

Theorem 10 (Lipman-Teissier)**.** *If (R, m) is pseudo-rational and $I \subseteq R$ is an ideal, then*

$$\overline{I^{\dim(R)+k-1}} \subseteq I^k \quad \forall k \geq 1.$$

Example 11 (Warmup)**.** Let $f, g, h \in \mathbb{C}[x, y]$. Show that $f^2g^2h^2 \in (f^3, g^3, h^3)$.
Solution to Warmup: Notice that

$$(f^2g^2h^2)^3 = f^6g^6h^6 = (f^3)^2(g^3)^2(h^3)^2 \in (f^3, g^3, h^3)^6.$$

This implies that $f^2g^2h^2 \in \overline{(f^3, g^3, h^3)^2}$. Since we are working with 3 elements, by the Briançon-Skoda theorem (or bounded by dimension 2 considerations), we have:

$$f^2g^2h^2 \in \overline{(f^3, g^3, h^3)^2} \subseteq^{\text{B.S.}}_{\dim 2} (f^3, g^3, h^3).$$

Example 12. Let $R = \frac{\mathbb{C}[x,y,z]}{(x^2+y^4+z^4)}$. Note that $\dim(R) = 2$. Then, $x^2 \in (y, z)^4 \implies x \in \overline{(y, z)^2}$, but $x \notin (y, z)$.
(Note: One can construct a surface singularity R so that $x \in \overline{(y, z)^2}$ but $x \notin (y, z)$).

1.3 Main Results

Theorem 13 (MMRS '25 - Theorem A)**.** *Let* $I \subseteq R$ *with* $\mu(I) = n$*.*

1. *If* R *is pseudo-rational, then* $\overline{I^{n+k-1}} \subseteq I^k$*.*
2. *If* R *is* F*-pure / Du Bois, then* $\overline{I^{n+k}} \subseteq I^k$ *for all* $k \geq 1$ *(in particular,* $\overline{I^{\dim(R)+1}} \subseteq I$*).*

Theorem 14 (MMRS '25 - Theorem B)**.** *Let* R *be excellent, reduced, with* $\dim(R) < \infty$*. Then there exists* N *depending on* R*, so that*

$$\overline{I^{N+k-1}} \subseteq I^k \quad \forall I \subseteq R,\ \forall k \geq 1.$$

1.4 Proofs

Lemma 15 (Kovacs)**.** *If* R *is pseudo-rational, then the natural map* $R \to R\pi_*\mathcal{O}_Y$ *splits in* $D(R)$*.*

Proof Sketch of Lemma. The pseudo-rationality condition gives an isomorphism of dualizing complexes:

$$\omega_R^\bullet = \omega_R \cong \pi_*\omega_Y \to R\pi_*\omega_Y \to R\pi_*\omega_Y^\bullet \xrightarrow{Tr} \omega_R^\bullet.$$

Applying $R\operatorname{Hom}(-, \omega_R^\bullet)$ to this gives the splitting:

$$R \leftarrow R\pi_*\mathcal{O}_Y \xleftarrow{\text{id}} R.$$

This means the composition is the identity. □

Proof of Theorem A. Assume $k = 1$. **Goal:** Show $\overline{I^n} \subseteq I$.

Say $I = (f_1, \ldots, f_n)$. Let $Y = \widetilde{Bl_I R} \xrightarrow{\pi} \operatorname{Spec}(R)$. Recall that $\overline{I^n} = \pi_*\mathcal{O}_Y(-nE)$, where $\mathcal{O}_Y(-E) = I\mathcal{O}_Y$.

Consider the Koszul complex of $\underline{f} = (f_1, \ldots, f_n)$ pulled back to Y:

$$\pi^* \operatorname{Kos}(\underline{f}, R) = \left(0 \to \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus n} \to \cdots \to \mathcal{O}_Y^{\oplus\binom{n}{2}} \to \mathcal{O}_Y^{\oplus n} \xrightarrow{(f_1,\ldots,f_n)} \mathcal{O}_Y \to 0\right).$$

Since each $f_i \in \mathcal{O}_Y(-E)$, we can view each f_i as a map $f_i \colon \mathcal{O}_Y(E) \to \mathcal{O}_Y$. This gives rise to an exact complex on Y:

$$0 \to \mathcal{O}_Y(-nE) \to \mathcal{O}_Y(-(n-1)E)^{\oplus n} \to \cdots \to \mathcal{O}_Y(-E)^{\oplus n} \xrightarrow{(f_1,\ldots,f_n)} \mathcal{O}_Y \to 0.$$

We can compare these two complexes via a canonical map of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y(-nE) \arrow[r] \arrow[d] & \mathcal{O}_Y(-(n-1)E)^{\oplus n} \arrow[r] \arrow[d] & \ldots \arrow[r] & \mathcal{O}_Y(-E)^{\oplus n} \arrow[r, "(f_1 \cdots f_n)"] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_Y \arrow[r] & \mathcal{O}_Y^{\oplus n} \arrow[r] & \ldots \arrow[r] & \mathcal{O}_Y^{\oplus n} \arrow[r, "(f_1 \cdots f_n)"] & \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

The canonical map $\mathcal{O}_Y(-nE) \to \mathcal{O}_Y \to \pi^* \operatorname{Kos}(\underline{f}, R)$ is 0 in the derived category $D(Y)$ (since it factors through an exact complex).

Apply $R\pi_*(-)$ to the map $\mathcal{O}_Y(-nE) \to \pi^* \operatorname{Kos}(\underline{f}, R)$:

$$\pi_*\mathcal{O}_Y(-nE) \to R\pi_*\mathcal{O}_Y(-nE) \to R\pi_*\pi^* \operatorname{Kos}(\underline{f}, R) \simeq \operatorname{Kos}(\underline{f}, R\pi_*\mathcal{O}_Y).$$

Using Kovacs's Lemma, the pseudo-rationality of R implies the splitting $R \to R\pi_*\mathcal{O}_Y \to R$, which induces a splitting of the corresponding Koszul complexes. We obtain the following commutative diagram:

$$\begin{tikzcd}
\overline{I^n} = \pi_*\mathcal{O}_Y(-nE) \arrow[r, "0"] \arrow[d] & \operatorname{Kos}(\underline{f}, R\pi_*\mathcal{O}_Y) \\
R = \pi_*\mathcal{O}_Y \arrow[r] & \operatorname{Kos}(\underline{f}, R) \arrow[u, "\text{splits}"']
\end{tikzcd}$$

This implies that the composition

$$\overline{I^n} \to R \to \operatorname{Kos}(\underline{f}, R)$$

is 0 in $D(R)$.

Taking the 0-th cohomology $H^0(-)$, we see that the natural map

$$\overline{I^n} \to R \to R/I$$

is 0. Therefore, the image of $\overline{I^n}$ in R/I is zero, which means exactly that

$$\overline{I^n} \subseteq I.$$

This completes the proof. □

2 A Gorenstein Criterion via the F-pure Threshold(Lecture By Suchitra Pande)

2.1 Preliminaries and Background

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring.

Definition 16 (Cohen–Macaulay)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. We say that R is *Cohen–Macaulay* if $\operatorname{depth}(R) = d$. Equivalently, some (equivalently every) system of parameters $x_1, \dots, x_d \in \mathfrak{m}$ is an R-regular sequence.

Definition 17 (Gorenstein)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. We say that R is *Gorenstein* if R is Cohen–Macaulay and, for some (equivalently every) system of parameters $x_1, \dots, x_d$, the Artinian quotient $R/(x_1, \dots, x_d)$ has one-dimensional socle:

$$\dim_k \operatorname{Ann}_{\mathfrak{m}}\big(R/(x_1, \dots, x_d)\big) = 1.$$

Equivalently, $\omega_R \cong R$.

Question: Criteria for Gorenstein rings?
Answers in characteristic $p > 0$.

Remark 18 (Notation and Setup)**.** Let $R \xrightarrow{F^e} R$ be the Frobenius ring homomorphism, where $r \mapsto r^{p^e}$. This makes $F^e_* R$ a new R-module.
For $F^e_* r \in F^e_* R$, the R-module action is $s \cdot F^e_* r = F^e_*(s^{p^e} r)$. This yields a sequence of R-modules $(F^e_* R)_{e \geq 1}$.

Assume $(R, \mathfrak{m})$ is a domain & is finite *(Explanation: F-finite).*

Remark 19 (Properties of F-regularity)**.**

1. **(Kunz):** R is regular $\iff$ $F^e_* R$ is free.
2. **(Hochster-Huneke):** If R is F-regular then R is normal & CM.

Fedder's Criterion: Provides a criterion for SFR.
Question: How to determine if a ring is Gorenstein?

Definition 20 (Hochster–Huneke)**.** Let R be an F-finite reduced ring of characteristic $p > 0$. We say that R is *strongly F-regular* if for every $c \in R$ not contained in any minimal prime, there exists $e \gg 0$ such that the map

$$R \longrightarrow F^e_* R, \qquad 1 \longmapsto F^e_* c$$

splits as a map of R-modules.

Definition 21 (F-pure threshold)**.** Let $(R, \mathfrak{m})$ be an F-finite local domain of characteristic $p > 0$, and assume R is F-pure. For each $e \geq 1$, define

$$\nu_e = \max\left\{ n \mid \exists f \in \mathfrak{m}^n \text{ such that } R \to F^e_* R,\ 1 \mapsto F^e_* f \text{ splits} \right\}.$$

Then

$$\operatorname{fpt}(R) = \lim_{e \to \infty} \frac{\nu_e}{p^e}.$$

Definition 22 (F-pure Thresholds)**.** The *F-pure threshold* of $(R, \mathfrak{m})$ requires the definition of ν_e. For $e \geq 1$,

$$\nu_e = \max\{n \mid 0 \neq f \in \mathfrak{m}^n \text{ s.t. } R \to F^e_* R \text{ splits } 1 \mapsto F^e_* f\}$$

Following Takagi-Watanabe, the F-pure threshold is defined as:

$$\operatorname{fpt}(R) = \lim_{e\to\infty} \frac{\nu_e}{p^e}$$

Example 23. Let $X = (x_{ij})_{m\times n}$ be a matrix of variables, $2 \leq r \leq m \leq n$. Consider I_r to be the ideal of $r \times r$ minors of X. Let $R = k[X]/I_r$.

- R is SFR $\implies$ R is CM.
- R is Gorenstein $\iff$ $m = n$.
- $\operatorname{fpt}(R) = m(r-1) \quad / \quad a(R) = -n(r-1)$.

2.2 Main Results and Conjectures

Gorenstein Criterion:

Conjecture 24 (Hirose-Watanabe-Yoshida)**.** Let S be a standard graded algebra over k. Let $\mathfrak{m}$ be the homogeneous maximal ideal, and $R = S_{\mathfrak{m}}$. Assume S is SFR.

1. $\operatorname{fpt}(R) \leq -a(R)$.
2. R is Gorenstein $\iff$ $\operatorname{fpt}(R) = -a(R)$.

Here, $a(R) = \max\{n \mid [H^d_{\mathfrak{m}}(S)]_n \neq 0\}$.

Remark 25 (Previous results)**.**

1. Hirose–Watanabe–Yoshida proved the toric case.
2. Chiba–Matsuda proved the Hibi ring case.
3. De Stefani–Núñez-Betancourt proved:
    - (i) $\operatorname{fpt}(R) \leq -a(R)$,
    - (ii) the "only if" direction of the conjecture, namely: if R is Gorenstein, then $\operatorname{fpt}(R) = -a(R)$.
4. Singh–Takagi–Varbaro proved the converse direction $\operatorname{fpt}(R) = -a(R) \Rightarrow R$ is Gorenstein under the additional assumption that the anti-canonical ring is finitely generated.

Theorem 26 (Main Theorem)**.** *Let S be a standard graded, strongly F-regular ring over an F-finite field of characteristic $p > 0$. Then*

$$\operatorname{fpt}_{\mathfrak{m}}(S) \leq -a(S),$$

and

$$S \text{ is Gorenstein} \iff \operatorname{fpt}_{\mathfrak{m}}(S) = -a(S).$$

2.3 Proofs

Lemma 27. *Let Y be projective, L ample on Y, $D \geq 0$ a Weil-divisor. Suppose we have $\mathbb{Q}$-divisors $\Delta_e \geq 0$, $\lambda_e \in \mathbb{Q}_{>0}$ such that*

$$D + \Delta_e \sim_{\mathbb{Q}} \lambda_e L$$

and $\lim_{e\to\infty} \lambda_e = 0$, then we necessarily have $D = 0$.

Proof Sketch of Main Theorem. Let $Y = \operatorname{Proj}(S)$ be a projective variety over $k = \overline{k}$.

Step i): Let $\mathcal{L} = \mathcal{O}_Y(1)$ be an ample line bundle, and ω_Y be the canonical sheaf on Y.

$$[H^d_{\mathfrak{m}}(S)]_n \cong H^d(Y, \mathcal{L}^n) \cong_{\text{Serre}} H^0(Y, \omega_Y \otimes \mathcal{L}^{-n})$$

(% Note: The first cohomology should technically be $H^{d-1}(Y, \mathcal{L}^n)$ if $\dim(S) = d$)
For $n \in \mathbb{Z}$:

$$a = -a(S) = \min\{n \mid H^0(\omega_Y \otimes \mathcal{L}^n) \neq 0\}$$

This implies $\exists$ effective divisor $D \geq 0$, $D \sim K_Y + aL$.

Step ii): Idea:

$$\left\{ \begin{array}{l} \text{homogeneous splittings} \\ \phi : F^e_* S \to S \\ \text{taking } f_n \in S_n \text{ to } 1 \end{array} \right\} \iff \operatorname{Hom}_{\mathcal{O}_Y}(F^e_*\mathcal{L}^n, \mathcal{O}_Y)$$

$$\operatorname{Hom}_{\mathcal{O}_Y}(F^e_*\mathcal{L}^n, \mathcal{O}_Y) \cong F^e_*(\omega_Y^{1-p^e} \otimes \mathcal{L}^{-n})$$

(Duality applied to F^e_*-)

Upshot: For ϕ mapping $F^e_*(f_n)$ to 1, with $f_n \in S_n$:

$$\rightsquigarrow H^0(Y, F^e_*(\omega_Y^{1-p^e} \otimes \mathcal{L}^{-n})) \underset{Y \xrightarrow{F^e} Y}{\cong} H^0(Y, \omega_Y^{1-p^e} \otimes \mathcal{L}^{-n})$$

$$\rightsquigarrow 0 \leq D_\phi \sim (1-p^e)K_Y - nL$$

$$\rightsquigarrow \Delta_\phi = \frac{1}{p^e-1} D_\phi \sim_{\mathbb{Q}} -K_Y - \frac{n}{p^e-1} L$$

Idea: F-splittings give us sections of $-K_Y$ (up to twists) but the a-invariant gives us sections of K_Y up to twists.

Step iii): Recall: $\exists f_e \in S_{\nu_e}$ and a splitting $\phi_e : F^e_* S \to S$ sending $F^e_* f_e \to 1$.

$$\rightsquigarrow 0 \leq \Delta_{\phi_e} \sim -K_Y - \frac{\nu_e}{p^e-1} L$$

$$\implies D + \Delta_{\phi_e} \sim \left(a - \frac{\nu_e}{p^e-1}\right) L$$

Step iv): On a projective variety Y, any effective divisor has non-negative degree.

$$\implies a \geq \frac{\nu_e}{p^e-1} \xrightarrow{e\to\infty} a \geq \operatorname{fpt}(S)$$

Using Lemma 27, since $D \geq 0$, $\Delta_{\phi_e} \geq 0$, and $\lambda_e = a - \frac{\nu_e}{p^e-1} \to 0$ as $e \to \infty$ (assuming equality in the conjecture $\operatorname{fpt}(R) = a$), we obtain:

$$0 = D \sim K_Y + aL \implies K_Y = -aL$$

$$\implies \omega_S \cong S(-a) \quad (\leftarrow \text{free module over } S)$$

$$\implies \omega_S \cong S \implies S \text{ is Gorenstein.}$$

□

Corrected proof sketch. Let $X = \operatorname{Proj}(S)$ and let L be the ample divisor corresponding to $\mathcal{O}_X(1)$. Write

$$a := -a(S).$$

Since $\dim S = d + 1$, we have

$$H^{d+1}_{\mathfrak{m}}(S) \cong \bigoplus_{r \in \mathbb{Z}} H^d(X, \mathcal{O}_X(rL)).$$

Hence

$$a = \min\{r \mid H^d(X, \mathcal{O}_X(-rL)) \neq 0\}.$$

By duality, this is equivalent to

$$a = \min\{r \mid H^0(X, \mathcal{O}_X(K_X + rL)) \neq 0\}.$$

Therefore there exists an effective Weil divisor $D \geq 0$ such that

$$D \sim K_X + aL.$$

Now let $f_e \in S_{n_e}$ be homogeneous and suppose there exists a splitting

$$\phi_e : F^e_* S \to S \qquad \text{with} \qquad \phi_e(F^e_* f_e) = 1.$$

Using duality for Frobenius on X, such a splitting gives an effective Weil divisor $D_e \geq 0$ satisfying

$$\frac{1}{p^e-1} D_e \sim_{\mathbb{Q}} -K_X - \frac{n_e}{p^e-1} L.$$

Adding this to $D \sim K_X + aL$ yields

$$D + \frac{1}{p^e-1} D_e \sim_{\mathbb{Q}} \left(a - \frac{n_e}{p^e-1}\right) L.$$

Since the left-hand side is effective, one obtains

$$\frac{n_e}{p^e-1} \leq a.$$

Passing to the limit gives

$$\operatorname{fpt}_{\mathfrak{m}}(S) \leq a = -a(S).$$

If equality holds, then one can choose n_e so that

$$a - \frac{n_e}{p^e - 1} \to 0.$$

Applying the divisor lemma to

$$D + \frac{1}{p^e - 1} D_e \sim_{\mathbb{Q}} \left(a - \frac{n_e}{p^e - 1} \right) L$$

forces $D = 0$. Hence

$$K_X \sim -aL.$$

Therefore the graded canonical module is isomorphic to a shift:

$$\omega_S \cong S(-a),$$

so S is quasi-Gorenstein. In the standard graded strongly F-regular setting, this gives the desired Gorenstein conclusion. $\square$

3 Koszul Duality for Toric Varieties (Lecture by Michael Brown)

3.1 Koszul Duality for Polynomial Rings

Let k be a field, and let A be an abelian group. Let $S = k[x_0, \ldots, x_n]$ be positively A-graded. That is, there exists a homomorphism $\theta : A \to \mathbb{Z}$ such that $\theta(\deg(s)) \geq 0$ for all homogeneous $s \in S$, and $\theta(\deg(s)) = 0 \iff s \in k$.

Let $E = \bigwedge_k (e_0, \ldots, e_n)$ be the exterior algebra. We give E an $(A \oplus \mathbb{Z})$-grading by defining:

$$\deg(e_i) = (-\deg(x_i), -1)$$

Definition 28. A **differential E-module** is an $(A \oplus \mathbb{Z})$-graded E-module D, equipped with a degree $(0, -1)$ endomorphism ∂ such that $\partial^2 = 0$.

Remark 29 (Explanation)**.** The grading ensures that the differential lowers the homological degree by 1 while preserving the internal A-grading. This setup naturally generalizes the standard Koszul complex differential.

Let $\mathrm{Com}(S)$ denote the category of complexes of graded S-modules, and let $\mathrm{DM}(E)$ denote the category of differential E-modules.

For a *graded S-module M*, define

$$F(M) = \bigoplus_{a \in A} M_a \otimes_k E$$

with differential

$$\partial_{F(M)}(m \otimes e) = \sum_{i=0}^{n} x_i m \otimes e_i e.$$

If instead $M^\bullet \in \mathrm{Com}(S)$ is a complex of graded S-modules with differential d_M, then the correct total differential on $F(M^\bullet)$ is

$$\partial(m \otimes e) = d_M(m) \otimes e + \sum_{i=0}^{n} x_i m \otimes e_i e,$$

up to the usual sign convention determined by the homological grading.

Example 30. 1. $F(k) = E$.

2. Let $S = k[x_0, x_1]$, with $\deg(x_0) = 1$ and $\deg(x_1) = 2$. The functor applied to S yields the complex $F(S)$:

$$\begin{tikzcd}
E \arrow[r, "e_0"] \arrow[dr, bend right, "e_1"'] & E(-1,0) \arrow[r, "e_0"] \arrow[dr, bend right, "e_1"'] & E(-2,0) \arrow[r, "e_0"] & E(-3,0) \arrow[r] & \cdots \simeq k \\
1 \arrow[r, mapsto] & x_0 \arrow[r, mapsto] & x_0^2, x_1 \arrow[r, mapsto] & x_0^3, x_0x_1 \ldots &
\end{tikzcd}$$

   Here, the standard basis elements track the degrees. The straight arrows represent multiplication by e_0 (corresponding to x_0), and the curved arrows represent multiplication by e_1 (corresponding to x_1).

Let $K_{\mathrm{DM}}(E) = \{\text{Free differential } E\text{-modules}\}/\text{homotopy}$.

Theorem 31. *The functor F induces an equivalence of derived categories:*

$$F : D(S) \xrightarrow{\sim} K_{DM}(E)$$

Furthermore, restricting to bounded complexes gives:

$$F : D^b(S) \xrightarrow{\sim} D^b_{DM}(E)$$

Note: $F : D(S) \overset{\sim}{\not\to} D_{\mathrm{DM}}(E)$.

Example 32. For $n = 0$, $F(S[x^{-1}])$ is exact.

Koszul Duality for Toric Varieties

Let X be a smooth projective toric variety. By standard toric geometry, X corresponds to the following algebraic data:

$$X \leftrightsquigarrow \begin{cases} \text{Cox ring } S = k[x_0, \dots, x_n], \text{ which is } \mathrm{Pic}(X)\text{-graded.} \\ \text{Irrelevant ideal } B \subseteq S, \text{ a square-free monomial ideal.} \end{cases}$$

Example 33 (Products of Projective Spaces)**.** Let $X = \mathbb{P}^m \times \mathbb{P}^n$. Then $\mathrm{Pic}(X) = \mathbb{Z}^2$. The Cox ring is $S = k[x_0, \dots, x_m, y_0, \dots, y_n]$.

The gradings are given by $\deg(x_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\deg(y_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The irrelevant ideal is $B = (x_0, \dots, x_m) \cap (y_0, \dots, y_n)$.

Example 34 (Hirzebruch Surface)**.** Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ with $r \geq 0 \implies \operatorname{Pic}(X) = \mathbb{Z}^2$.
The Cox ring is $S = k[x_0, x_1, y_0, y_1]$.
The gradings are:

$$\deg(x_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \deg(y_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \deg(y_1) = \begin{pmatrix} -r \\ 1 \end{pmatrix}$$

The irrelevant ideal is $B = (x_0, x_1) \cap (y_0, y_1)$.

Let H be an ample line bundle on X. This induces a homomorphism θ:

$$\theta : \operatorname{Pic}(X) \to \mathbb{Z}, \quad \theta(D) = \langle D, H^{\dim X - 1} \rangle$$

Theorem 35 (BGG 1978, Buchweitz 1986)**.** *Let $E = \bigwedge_k(e_0, \ldots, e_n)$ with $\deg(e_i) = -1$. We have the following equivalences:*

$$D^b(\mathbb{P}^n) \simeq K^{ex}(E) = \{\textit{exact complexes of f.g. graded free } E\textit{-modules}\}$$

Moreover, the singularity category of E is given by:

$$\textit{singularity category} \simeq D^b(E)/\textit{Perf}(E)$$

Remark 36 (Explanation)**.** The Bernstein-Gelfand-Gelfand (BGG) correspondence fundamentally relates the derived category of coherent sheaves on projective space $D^b(\mathbb{P}^n)$ with modules over the exterior algebra E. Specifically, $D^b(\mathbb{P}^n) \simeq D^b(S)/D^b_M(S)$, where M is the irrelevant maximal ideal.

Furthermore, via Eisenbud-Fløystad-Schreyer 2003 (sheaf cohomology algorithm over $\mathbb{P}^n$), we look at extending this to X. For a general smooth projective toric variety X, let:

$$E = \bigwedge_k(e_0, \ldots, e_n), \quad \text{which is } (\operatorname{Pic}(X) \oplus \mathbb{Z})\text{-graded, with } \deg(e_i) = (-\deg(x_i), -1).$$

We want to understand the relation:

$$D^b(X) \simeq D^b(S)/D^b_B(S) \simeq D^b_{\mathrm{DM}}(E)/\boxed{?}$$

Definition 37. A subset $I \subseteq \{0, \ldots, n\}$ is **irrelevant** if $B \subseteq (x_i : i \in I)$.
For any subset $I \subseteq \{0, \ldots, n\}$, we define the ideal $\Theta_I \subseteq E$ as:

$$\Theta_I = (e_i \in E : i \notin I)$$

We define the category of exact differential modules with respect to these irrelevant ideals:

$$K^{ex}_{\mathrm{DM}}(E) = \{F \in K_{\mathrm{DM}}(E) \mid F \otimes_E E/\Theta_I \text{ is exact } \forall \text{ irrelevant } I\}$$

Let $R\Gamma_*$ denote the total derived global sections functor:

$$R\Gamma_*(\mathcal{F}) = \bigoplus_{d \in \operatorname{Pic}(X)} R\Gamma(X, \mathcal{F}(d))$$

Theorem 38 (Brown-Erman)**.** *Let X be a smooth projective toric variety. Then the composition*

$$F \circ R\Gamma_* : D(X) \to K^{ex}_{DM}(E)$$

is an equivalence of categories.